SELF-RECIPROCAL FUNCTIONS, POWERS OF THE RIEMANN ZETA FUNCTION AND MODULAR-TYPE TRANSFORMATIONS

ATUL DIXIT AND VICTOR H. MOLL

Abstract. The classical transformation of Jacobi's theta function admits a simple proof by producing an integral representation that yields this invariance apparent. This idea seems to have first appeared in the work of S. Ramanujan. Several examples of this idea have been produced by Koshlyakov, Ferrar, Guinand, Ramanujan and others. A unifying procedure to analyze these examples and natural generalizations is presented.

1. Introduction

In his approach to the theory of elliptic functions, C. G. J. Jacobi [17] introduced his classical theta function

$$\vartheta_3(x,\omega) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x + \pi i n^2 \omega} \tag{1.1}$$

and three other similar functions ϑ_1, ϑ_2, ϑ_4. These are entire functions of $x \in \mathbb{C}$, so they cannot be doubly-periodic, but every elliptic function can be written in terms of them. The transformations of the so-called *null-values* $\vartheta_1'(0,\omega)$, $\vartheta_j(0,\omega)$ for $2 \leq j \leq 4$ under the modular group $PSL(2,\mathbb{Z})$ are of intrinsic interest. Jacobi proved that the transformation of $\vartheta_3(0,it)$ for Re $t>0$, yields the pretty identity

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t}. \tag{1.2}$$

The reader will find details in [24, Chapter 3]. This identity may be written in a more symmetric form as

$$\sqrt{\alpha}\left(\frac{1}{2\alpha} - \sum_{n=1}^{\infty} e^{-\pi\alpha^2 n^2}\right) = \sqrt{\beta}\left(\frac{1}{2\beta} - \sum_{n=1}^{\infty} e^{-\pi\beta^2 n^2}\right), \tag{1.3}$$

with α, $\beta > 0$ and $\alpha\beta = 1$. These type of identities are called modular-type transformations. A procedure to establish the identity (1.3) is to produce an

Date: December 4, 2013.
2010 *Mathematics Subject Classification.* Primary 11M06, Secondary 33C05.
Key words and phrases. Riemann Ξ-function, Bessel functions, integral identities, Koshlyakov, Ramanujan.

integral representation of one of the sides that is invariant under $\alpha \to 1/\alpha$. To see this, define

$$\Xi(t) = \xi\left(\tfrac{1}{2} + it\right), \tag{1.4}$$

where ξ is the Riemann ξ-function

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s), \tag{1.5}$$

with

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt \text{ and } \zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \tag{1.6}$$

the gamma and Riemann zeta function, respectively. The definition of $\Gamma(s)$ in (1.6) is valid for $\operatorname{Re} s > 0$ and that for $\zeta(s)$ is valid for $\operatorname{Re} s > 1$. These functions are then extended to $\mathbb{C}$ by meromorphic continuation, with poles at $s = 0, -1, -2, \ldots$ for $\Gamma(s)$, and at $s = 1$ for $\zeta(s)$. The integral evaluation [29, p. 36]

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(t/2)}{1+t^2}\cos(\tfrac{1}{2}t\log\alpha)\,dt = \sqrt{\alpha}\left(\frac{1}{2\alpha} - \sum_{n=1}^\infty e^{-\pi\alpha^2n^2}\right) \tag{1.7}$$

and the obvious invariance of the left-hand side under $\alpha \to 1/\alpha$ gives (1.3).

The goal of this paper is to study, in a unified manner, a variety of integrals of the form

$$I(f,z;\alpha) = \int_0^\infty f\left(z, \tfrac{t}{2}\right)\Xi\left(\tfrac{t-iz}{2}\right)\Xi\left(\tfrac{t+iz}{2}\right)\cos\left(\tfrac{1}{2}t\,\log\alpha\right)\,dt, \tag{1.8}$$

where $f(z,t)$ is an even function of t of the form

$$f(z,t) = \phi(z,it)\phi(z,-it), \tag{1.9}$$

with ϕ analytic as a function of $t \in \mathbb{R}$ and $z \in \mathbb{C}$. The integral (1.8) extends

$$I(f;\alpha) = \int_0^\infty f\left(\tfrac{t}{2}\right)\Xi\left(\tfrac{t}{2}\right)\cos\left(\tfrac{1}{2}t\,\log\alpha\right)\,dt, \tag{1.10}$$

with

$$f(t) = \varphi(it)\varphi(-it), \tag{1.11}$$

and φ is analytic in t. Particular examples of this latter integral were studied by Ramanujan, Hardy, Koshlyakov and Ferrar (see also [33, p. 35]), whereas Ramanujan [31] was the first person to study an integral of the type in (1.8). It is clear that (1.8) and (1.10) are invariant under $\alpha \to 1/\alpha$. An alternative expression (series or integral) of $I(f,\alpha)$ then yields identities similar to (1.3). These are called *modular-type transformations*. It is easy to extend these identities by analytic continuation to $\alpha, \beta \in \Omega \subset \mathbb{C}$, with $\mathbb{R} \subset \Omega$. The classical example in (1.3) corresponds to taking $f(t) = 1/(1+4t^2)$ in (1.10).

The identities obtained here have the form

(1.12)
$$F(z,\alpha) = F(z,\beta) = \int_0^\infty f\left(z, \tfrac{t}{2}\right) \Xi\left(\tfrac{t-iz}{2}\right) \Xi\left(\tfrac{t+iz}{2}\right) \cos\left(\tfrac{1}{2}t\,\log\alpha\right)\,dt,$$

for a suitably chosen function f. The parameters α and β are positive and satisfy $\alpha\beta = 1$. Naturally, if the identity

$$F(z,\alpha) = \int_0^\infty G(z,t)\cos\left(\tfrac{1}{2}t\,\log\alpha\right)\,dt, \tag{1.13}$$

has been established, it follows immediately that $F(z,\alpha) = F(z,\beta)$.

Section 2 introduces a technique for studying integrals in (1.10) through an example containing $K_0(x)$, the modified Bessel function of order 0. Section 3 describes an example found in Ramanujan's work and links it to another of his integrals. This illustrates the fundamental idea of this paper. In Section 4, some classical formulas are generalized by the introduction of a new parameter z. This section states the results with the proofs presented in Section 5. Generalizations of some results of Koshlyakov, particularly dealing with his function $\Omega(x)$ (see (6.1) below), are stated in Section 6. Section 7 discusses advantages of our methods over those of A.P. Guinand and C. Nasim. Finally, the last section describes future directions of the work discussed here.

2. An illustrative example

This section presents a general technique to evaluate integrals of the type (1.10). This is illustrated with an example established by N. S. Koshlyakov [18]. The proof given here follows [6] and [11].

The function

$$f(t) = \frac{4\Xi(t)}{(\frac{1}{4}+t^2)^2} \tag{2.1}$$

admits a factorization in the form (1.11) with

$$\varphi(s) = \frac{2\sqrt{\xi\left(\frac{1}{2}-s\right)}}{\left(\frac{1}{2}+s\right)\left(\frac{1}{2}-s\right)}. \tag{2.2}$$

It is known [33, p. 35] that

(2.3)
$$\int_0^\infty f\left(\tfrac{t}{2}\right)\Xi\left(\tfrac{t}{2}\right)\cos\left(\tfrac{1}{2}t\log\alpha\right)\,dt = \frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \varphi\left(s-\tfrac{1}{2}\right)\varphi\left(\tfrac{1}{2}-s\right)\xi(s)\alpha^s\,ds$$

Using (2.1) and (2.2), this yields

$$\int_0^\infty \frac{64\,\Xi^2\left(\frac{t}{2}\right)\cos(\frac{1}{2}t\log\alpha)}{(1+t^2)^2}\,dt = \frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma^2\left(\tfrac{s}{2}\right)\zeta^2(s)\left(\frac{\alpha}{\pi}\right)^s\,ds. \tag{2.4}$$

To evaluate the contour integral on the right-hand side, square the functional equation for $\zeta(s)$

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{2.5}$$

and recall the Dirichlet series

$$\zeta^2(s) = \sum_{n=1}^{\infty}\frac{d(n)}{n^s}, \tag{2.6}$$

where $d(n)$ is the number of divisors of n; see [1, page 229]. The expansion (2.6) is valid for $\operatorname{Re} s > 1$, so it is necessary to move the line of integration in (2.4) to $\operatorname{Re} s = 1+\delta$, for some $\delta > 0$. This process captures a pole of the integrand at $s = 1$, with residue

$$\lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\left(\frac{\alpha}{\pi}\right)^s\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\right) = \alpha\left(\gamma - \log\left(\frac{4\pi}{\alpha}\right)\right), \tag{2.7}$$

where $\gamma = -\Gamma'(1)$ is the Euler's constant. Then

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s ds = \sum_{m=1}^{\infty}d(m)\int_{1+\delta-i\infty}^{1+\delta+i\infty}\Gamma^2\left(\frac{s}{2}\right)\left(\frac{\pi m}{y}\right)^{-s} ds.$$

The integral on the right-hand side appears in [28, p.115, formula 11.1]

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2^{s-2}a^{-s}\Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)x^{-s}\, ds = K_\nu(ax), \tag{2.8}$$

valid for Re $s > \pm$ Re ν and $a > 0$. Here $K_\nu(w)$ is the modified Bessel function of order ν, defined by [12, p. 928, formula 8.485]

$$K_\nu(w) = \frac{\pi}{2}\frac{(I_{-\nu}(w) - I_\nu(w))}{\sin\pi\nu} \tag{2.9}$$

where [12, p. 911, formula 8.406, nos. 1-2]

$$I_\nu(w) = \begin{cases} e^{-\frac{\pi}{2}\nu i}J_\nu(e^{\frac{\pi i}{2}}w) & \text{if } -\pi < \arg w \leq \frac{\pi}{2}, \\ e^{\frac{3}{2}\pi\nu i}J_\nu(e^{-\frac{3}{2}\pi i}w) & \text{if } \frac{\pi}{2} < \arg w \leq \pi, \end{cases} \tag{2.10}$$

and [12, p. 910, formula 8.402]

$$J_\nu(w) := \left(\frac{w}{2}\right)^{\nu}\sum_{n=0}^{\infty}\frac{(-w^2/4)^n}{\Gamma(\nu+1+n)n!} \tag{2.11}$$

is the Bessel function of the first kind. The result is an integral of type (1.10), stated here for $\nu = 0$.

Theorem 2.1. *Let $d(n)$ be the number of divisors of $n \in \mathbb{N}$, γ the Euler constant and $K_0(w)$ the modified Bessel function of order* 0. *Then, for*

$\alpha, \beta > 0, \alpha\beta = 1,$

(2.12)

$$-\frac{32}{\pi}\int_0^\infty \frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2}{(1+t^2)^2}\cos(\tfrac{1}{2}t\log\alpha)\,dt = \sqrt{\alpha}\left(\frac{\gamma-\log(4\pi\alpha)}{\alpha} - 4\sum_{n=1}^\infty d(n)K_0(2\pi n\alpha)\right)$$
$$= \sqrt{\beta}\left(\frac{\gamma-\log(4\pi\beta)}{\beta} - 4\sum_{n=1}^\infty d(n)K_0(2\pi n\beta)\right).$$

3. An example from Ramanujan

The success of the method described in Section 2 depends on the appropriate choice of the function $\varphi(t)$ in (1.11) and the ability to evaluate, or at least transform, the resulting integral $I(f;\alpha)$. This section presents a second example that illustrates this point of view.

Take

$$\varphi(t) = \Gamma\left(\frac{2t-1}{4}\right) \tag{3.1}$$

and

$$f(t) = \Gamma\left(\frac{2it-1}{4}\right)\Gamma\left(\frac{-2it-1}{4}\right) = \left|\Gamma\left(\frac{2it-1}{4}\right)\right|^2, \tag{3.2}$$

in view of $\overline{\Gamma(z)} = \Gamma(\bar{z})$. The integral (1.10) becomes

$$\text{(3.3)}\; I(f;\alpha) = \int_0^\infty \left|\Gamma\left(\frac{it-1}{4}\right)\right|^2 \Xi\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)dt$$
$$= \int_0^\infty \Gamma\left(\frac{it-1}{4}\right)\Gamma\left(\frac{-it-1}{4}\right)\Xi\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)dt.$$

The functional equation $\Gamma(z+1) = z\Gamma(z)$, with $z = (it-1)/4$, now gives a new representation of (3.3):

$$I(f;\alpha) = 16\int_0^\infty \Gamma\left(\frac{3+it}{4}\right)\Gamma\left(\frac{3-it}{4}\right)\frac{\Xi\left(\frac{t}{2}\right)}{1+t^2}\cos\left(\frac{1}{2}t\log\alpha\right)dt. \tag{3.4}$$

This integral was transformed by S. Ramanujan [31, Equation (13)].

Theorem 3.1 (Ramanujan)**.** *The identity*

$$\int_0^\infty \Gamma\left(\frac{3+it}{4}\right)\Gamma\left(\frac{3-it}{4}\right)\frac{\Xi\left(\frac{t}{2}\right)}{1+t^2}\cos\left(\frac{1}{2}t\log\alpha\right)dt = \tag{3.5}$$
$$\pi^{5/2}\alpha^{3/2}\int_0^\infty xe^{-\pi\alpha^2x^2}\left(\frac{1}{2\pi x} - \frac{1}{e^{2\pi x}-1}\right)dx$$

holds for $\alpha > 0$.

This evaluation generates a modular-type transformation, which naturally leads to a beautiful identity among definite integrals.

Corollary 3.2. *Let* α, $\beta > 0$ *with* $\alpha\beta = 1$. *Then*

$$\alpha^{3/2}\int_0^\infty xe^{-\pi\alpha^2x^2}\left(\frac{1}{2\pi x}-\frac{1}{e^{2\pi x}-1}\right)dx = \beta^{3/2}\int_0^\infty xe^{-\pi\beta^2x^2}\left(\frac{1}{2\pi x}-\frac{1}{e^{2\pi x}-1}\right)dx.$$

Note 3.3. Koshlyakov example (2.12) is obtained by squaring $\Xi(t/2)/(1+t^2)$, which is part of the integrand in (1.7) appearing in the proof of the classical theta function identity. Thus is it natural to consider the integral

$$\int_0^\infty \Gamma\left(\frac{3+it}{4}\right)\Gamma\left(\frac{3-it}{4}\right)\left(\frac{\Xi\left(\frac{t}{2}\right)}{1+t^2}\right)^2\cos\left(\tfrac{1}{2}t\log\alpha\right)dt \tag{3.6}$$

as a variation of (3.5). Up to a constant factor, this may be expressed as

$$\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\frac{\Xi^2\left(\frac{t}{2}\right)}{1+t^2}\cos\left(\tfrac{1}{2}t\log\alpha\right)dt, \tag{3.7}$$

which is exactly the integral presented by S. Ramanujan at the end of his paper [31, Equation (22)]. Thus the idea of squaring the functional equation of $\zeta(s)$ to produce new transformations is implicit in the work of Ramanujan, much before Koshlyakov. In his Lost Notebook [32], Ramanujan gives the following beautiful modular-type transformation resulting from this integral.

Theorem 3.4. *Let*

$$\lambda(x) = \psi(x) + \frac{1}{2x} - \log x, \tag{3.8}$$

where

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{m=0}^\infty\left(\frac{1}{m+x}-\frac{1}{m+1}\right) \tag{3.9}$$

is the logarithmic derivative of the Gamma function. If α *and* β *are positive numbers such that* $\alpha\beta = 1$, *then*

$$\sqrt{\alpha}\left(\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{k=1}^\infty\lambda(k\alpha)\right) = \sqrt{\beta}\left(\frac{\gamma-\log(2\pi\beta)}{2\beta}+\sum_{k=1}^\infty\lambda(k\beta)\right)$$
$$= -\frac{1}{\pi^{3/2}}\int_0^\infty\left|\Gamma\left(\frac{-1+it}{4}\right)\right|^2\Xi^2\left(\frac{t}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)dt}{1+t^2}.$$

4. Some parametric generalizations. Statements of the results

This section contains several new modular-type transformations. The results are stated in this section with their proofs postponed to Section 5.

There are (at least) two approaches towards obtaining these transformations. One is presented in Theorems 5.3 and 5.5 and the corollaries following them. The other method involves the evaluation of integrals involving the Riemann Ξ-function, illustrated by (4.1) below. Given a modular-type transformation, the use of Parseval's identity (5.4) produces an integral involving the Riemann Ξ-function. Conversely, having a representation for an aforementioned integral involving the Riemann Ξ-function, obtained by residue calculus and Mellin transforms, the obvious invariance of these integrals under $\alpha \mapsto 1/\alpha$, provides a modular-type transformation. Examples of these two methods are presented in the proofs of the main results.

Implicit in both methods is the idea of squaring the functional equation of $\zeta(s)$. In the second method, this is reflected in the term $\Xi\left(\frac{t}{2}\right)/(1+t^2)$, present in the integrands. This is generalized by the inclusion a new parameter z.

In his work related to (3.7), Ramanujan [31] considered the generalization

$$\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right) \times \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos(\frac{1}{2}t\log\alpha)\,dt}{(t^2+(z+1)^2} \tag{4.1}$$

of (3.7). He provided alternative integral representations valid in different regions in the complex plane. Modular-type transformations for the above integral, which involve the Hurwitz zeta function, appear in [7, 8]. It should be mentioned here that Ramanujan had not only discovered Koshlyakov's formula (2.12) about 10 years before, but had also generalized it. Details appear in [3]. The generalization presented below, was later rediscovered by A. P. Guinand [14] and is rephrased in a symmetric form in the theorem below.

Theorem 4.1. *Let $K_\nu(s)$ denote the modified Bessel function of order ν and let $\sigma_s(n) = \sum_{d|n} d^s$. For $-1 < \operatorname{Re} z < 1$ define*

$$\omega(z,\alpha) = \left(\frac{\alpha}{\pi}\right)^{z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z). \tag{4.2}$$

Then, if α, $\beta > 0$ with $\alpha\beta = 1$,

$$\frac{1}{\sqrt{\alpha}}\left(\omega(z,\alpha)+\omega(-z,\alpha)-4\alpha\sum_{n=1}^\infty \sigma_{-z}(n)n^{z/2}K_{z/2}(2n\pi\alpha)\right) = \frac{1}{\sqrt{\beta}}\left(\omega(z,\beta)+\omega(-z,\beta)-4\beta\sum_{n=1}^\infty \sigma_{-z}(n)n^{z/2}K_{z/2}(2n\pi\beta)\right). \tag{4.3}$$

Note 4.2. The symmetry in α and β suggests the existence of an integral involving the Riemann Ξ-function similar to (4.1), which generalizes that giving rise to Koshlyakov's formula (2.12). This integral, found in [8], is

$$-\frac{32}{\pi}\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)\,dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}. \tag{4.4}$$

The term $\Xi^2(t/2)/(1+t^2)^2$ in (2.12), which resulted from squaring part of the integrand in (1.7), is now generalized to

$$\frac{\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}$$

in (4.4). The integral in (4.1) can also be rewritten to contain the above expression. To the best of our knowledge, (4.4) is the only other integral of this type, besides Ramanujan's (4.1), that has been studied up to now. Several new integrals of this type are presented next.

Note 4.3. As mentioned above, Koshlyakov made use of the idea of squaring the functional equation for $\zeta(s)$ in order to obtain some new transformations through the existing ones. These include Hardy's formula [15], rephrased in a compact form given by Koshlyakov [22, Equations (14), (20)]:

(4.5)

$$\sqrt{\alpha}\int_0^\infty e^{-\pi\alpha^2x^2}\left(\psi(x+1)-\log x\right)\,dx=\sqrt{\beta}\int_0^\infty e^{-\pi\beta^2x^2}\left(\psi(x+1)-\log x\right)\,dx$$
$$=2\int_0^\infty\frac{\Xi(t/2)}{1+t^2}\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{\cosh\frac{1}{2}\pi t}\,dt.$$

Koshlyakov [22, Equations (36), (40)] squared the term $\Xi(t/2)/(1+t^2)$ in the integral on the extreme right above and obtained the following result[1].

Theorem 4.4. *Define*

$$\Lambda(x)=\frac{\pi^2}{6}+\gamma^2-2\gamma_1+2\gamma\log x+\frac{1}{2}\log^2x+\sum_{n=1}^\infty d(n)\left(\frac{1}{x+n}-\frac{1}{n}\right),\tag{4.6}$$

where γ_1 *is the Stieltjes constant defined by*

$$\gamma_1=\lim_{m\to\infty}\left(\sum_{k=1}^m\frac{\log k}{k}-\frac{(\log m)^2}{2}\right).$$

Then, for $\alpha,\beta>0,\alpha\beta=1$,

$$\sqrt{\alpha}\int_0^\infty K_0(2\pi\alpha x)\Lambda(x)\,dx=\sqrt{\beta}\int_0^\infty K_0(2\pi\beta x)\Lambda(x)\,dx$$
$$=8\int_0^\infty\frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2}{(1+t^2)^2}\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{\cosh\left(\frac{1}{2}\pi t\right)}\,dt.\tag{4.7}$$

In his work, Koshlyakov did not consider one variable generalizations of the type in (4.1) and (4.4) of any of his formulas. The next result presents a generalization of (4.7) where $\Xi^2(t/2)/(1+t^2)^2$ is generalized to

$$\frac{\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}.$$

[1] In equation (40) in Koshlyakov's paper, there is a minus sign missing in front on the right-hand side.

Furthermore, the reciprocal of $\cosh\frac{1}{2}\pi t$ is replaced by a product of four gamma functions.

Theorem 4.5. *Assume $-1 < \operatorname{Re} z < 1$ and let γ, γ_1 and $K_\nu(z)$ be as before. Define*

(4.8)
$$\Lambda(x,z) = x^{z/2}\Gamma(1+z)\left\{\frac{x^{-z}}{-z}\zeta(1-z) + (2\gamma + \log x + \psi(1+z))\zeta(1+z) + \zeta'(1+z) + \sum_{n=1}^{\infty}\sigma_{-z}(n)\left(\frac{n^z}{(n+x)^{z+1}} - \frac{1}{n}\right)\right\}.$$

Then, for $\alpha, \beta > 0$ and $\alpha\beta = 1$,

$$\sqrt{\alpha}\int_0^\infty K_{z/2}(2\pi\alpha x)\Lambda(x,z)dx = \sqrt{\beta}\int_0^\infty K_{z/2}(2\pi\beta x)\Lambda(x,z)dx \tag{4.9}$$
$$= \frac{2^{z+2}}{\pi^2}\int_0^\infty \Gamma\left(\frac{z+3+it}{4}\right)\Gamma\left(\frac{z+3-it}{4}\right)\Gamma\left(\frac{z+1+it}{4}\right) \Gamma\left(\frac{z+1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)\,dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}.$$

Theorem 4.6. *Assume $-1 < \operatorname{Re} z < 1$. Let*

$$\mu(x,z) = \frac{\Gamma(1+z)\zeta(1+z)x^{-1-z/2}}{(2\pi)^{2+z}} \tag{4.10}$$

and define

$$\Phi(x,z) = 2\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_z(4\pi\sqrt{nx}) - \mu(x,z) - \mu(x,-z). \tag{4.11}$$

Then, for $\alpha, \beta > 0$ and $\alpha\beta = 1$,

$$\sqrt{\alpha^3}\int_0^\infty xK_{z/2}(2\pi\alpha x)\Phi(x,z)dx = \sqrt{\beta^3}\int_0^\infty xK_{z/2}(2\pi\beta x)\Phi(x,z)dx \tag{4.12}$$
$$= \frac{2}{\pi^4}\int_0^\infty \Gamma\left(\frac{z+3+it}{4}\right)\Gamma\left(\frac{z+3-it}{4}\right)\Gamma\left(\frac{-z+3+it}{4}\right) \times\Gamma\left(\frac{-z+3-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)\,dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}.$$

The series $\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}K_z(4\pi\sqrt{nx})$, along with some of its special cases, is treated at length in [5]. The special case $z = 0$ of the above theorem is interesting enough to be singled out.

Corollary 4.7. *Let $\alpha, \beta > 0$ and $\alpha\beta = 1$. Then*

$$\sqrt{\alpha^3}\int_0^\infty xK_0(2\pi\alpha x)\left(2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{nx})+\frac{\log(4\pi^2x)}{4\pi^2x}\right)dx= \tag{4.13}$$
$$=\sqrt{\beta^3}\int_0^\infty xK_0(2\pi\beta x)\left(2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{nx})+\frac{\log(4\pi^2x)}{4\pi^2x}\right)dx$$
$$=\frac{1}{128\pi^4}\int_0^\infty \Gamma^2\left(\frac{-1-it}{4}\right)\Gamma^2\left(\frac{-1+it}{4}\right)\Xi^2\left(\frac{t}{2}\right)\cos\left(\tfrac{1}{2}t\log\alpha\right)dt.$$

Using Voronoï's identity [34, Equations (5), (6)]

$$2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{nx})=\frac{a}{\pi^2}\sum_{n=1}^\infty\frac{d(n)\log(x/n)}{x^2-n^2}-\frac{\gamma}{2}-\left(\frac{1}{4}+\frac{1}{4\pi^2x}\right)\log x-\frac{\log 2\pi}{2\pi^2x}, \tag{4.14}$$

the above transformation is written in an equivalent different form. It should be pointed out that these identities also appear in Ramanujan's Lost Notebook [32, p. 254] (see also [3, Equation (4.1)]).

This is now rephrased into yet another form. This provides an interesting modular-type transformation between two double integrals as shown below.

Theorem 4.8. *Let $J_\nu(x)$ denote the Bessel function of the first kind of order ν. For $\alpha, \beta > 0$, $\alpha\beta = 1$, we have*

$$\sqrt{\alpha}\int_0^\infty\int_0^\infty\frac{y}{(y^2+t^2)^{3/2}}\left(J_0(2\alpha y)+\frac{4\pi t}{e^{2\pi t}-1}\left(\frac{1}{e^{2\pi\alpha y}-1}-\frac{1}{2\pi\alpha y}\right)\right)dy\,dt$$
$$=\sqrt{\beta}\int_0^\infty\int_0^\infty\frac{y}{(y^2+t^2)^{3/2}}\left(J_0(2\beta y)+\frac{4\pi t}{e^{2\pi t}-1}\left(\frac{1}{e^{2\pi\beta y}-1}-\frac{1}{2\pi\beta y}\right)\right)dy\,dt$$
$$=\frac{1}{8\pi^2}\int_0^\infty\Gamma^2\left(\frac{-1-it}{4}\right)\Gamma^2\left(\frac{-1+it}{4}\right)\Xi^2\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)dt. \tag{4.15}$$

An identity of the same type as (3.5) was established by W. L. Ferrar [11], written below in the form provided in [9]. For $\alpha, \beta > 0, \alpha\beta = 1$, we have:

$$\sqrt{\alpha}\int_0^\infty e^{-\pi\alpha^2x^2}\left(\sum_{n=1}^\infty K_0(2\pi nx)-\frac{1}{4x}\right)dx \tag{4.16}$$
$$=\sqrt{\beta}\int_0^\infty e^{-\pi\beta^2x^2}\left(\sum_{n=1}^\infty K_0(2\pi nx)-\frac{1}{4x}\right)dx$$
$$=\frac{-1}{2\pi^{3/2}}\int_0^\infty\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\frac{\cos(\frac{1}{2}t\log\alpha)\,dt}{1+t^2}.$$

The next theorem gives a generalization of Ferrar's result (4.16).

Theorem 4.9. *Assume* $-1 < \operatorname{Re} z < 1$ *and define*

(4.17)
$$\mathfrak{F}(x,z) = x^{z/2}\Gamma\left(\frac{1+z}{2}\right)\left(\left[3\gamma + 2\log x + \psi\left(\tfrac{1}{2}(1+z)\right)\right]\zeta(1+z) + 2\zeta'(1+z)\right)$$
$$-\sqrt{\pi}\,\Gamma\left(\frac{z}{2}\right)\zeta(1-z)x^{-z/2} + 2x^{z/2}\Gamma\left(\frac{1+z}{2}\right)\sum_{n=1}^{\infty}\sigma_{-z}(n)\left(\frac{n^z}{(x^2+n^2)^{\frac{1+z}{2}}} - \frac{1}{n}\right).$$

Then, for $\alpha, \beta > 0$, $\alpha\beta = 1$,

$$\sqrt{\alpha}\int_0^{\infty} K_{z/2}(2\pi\alpha x)\mathfrak{F}(x,z)dx = \sqrt{\beta}\int_0^{\infty} K_{z/2}(2\pi\beta x)\mathfrak{F}(x,z)dx \tag{4.18}$$
$$= \frac{8}{\pi}\int_0^{\infty}\Gamma\left(\frac{z+1+it}{4}\right)\Gamma\left(\frac{z+1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)$$
$$\times\frac{\cos(\frac{1}{2}t\log\alpha)dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}.$$

Note 4.10. The special value $z = 0$ gives the identity

$$\sqrt{\alpha}\int_0^{\infty} K_0(2\pi\alpha x)\mathfrak{F}(x,0)dx = \sqrt{\beta}\int_0^{\infty} K_0(2\pi\beta x)\mathfrak{F}(x,0)dx$$
$$= \frac{8}{\pi}\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\frac{\Xi^2(t/2)}{(1+t^2)^2}\cos(\tfrac{1}{2}t\log\alpha)\,dt,$$

where the function $\mathfrak{F}(x,0)$ has the explicit form

$$\mathfrak{F}(x,0) = \frac{\pi^2}{24} + \frac{\gamma^2}{2} - \gamma\log 2 + \frac{\log^2 2}{4} - \gamma_1 + \gamma\log x + \frac{\log x}{4}\log\left(\frac{x}{4}\right) + \frac{1}{2}\sum_{n=1}^{\infty} d(n)\left(\frac{1}{\sqrt{n^2+x^2}} - \frac{1}{n}\right), \tag{4.19}$$

and where γ, γ_1 and $d(n)$ are as before.

5. Proofs

The Mellin transform

$$F(s) := \mathfrak{M}[f;s] = \int_0^{\infty} x^{s-1}f(x)dx \tag{5.1}$$

is defined for a locally integrable function f. The existence of F depends on the asymptotic behavior of f at $x = 0$ and ∞. In detail, if

$$f(x) = \begin{cases} O(x^{-a-\varepsilon}) & \text{as } x \to 0^+, \\ O(x^{-b+\varepsilon}) & \text{as } x \to +\infty, \end{cases} \tag{5.2}$$

where $\varepsilon > 0$ and $a < b$, then $F(s)$ is an analytic function in the strip $a < \operatorname{Re} s < b$. The properties of the Mellin transform used here include the inversion formula

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}F(s)ds \tag{5.3}$$

and Parseval's identity

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(1-s)G(s)ds, \tag{5.4}$$

where the vertical line $\mathrm{Re}\, s = c$ lies in the common strip of analyticity of the Mellin transforms $F(1-s)$ and $G(s)$. See [29, p. 83] for the conditions on the validity of this formula. A variant of this identity is

$$\int_0^\infty f(t)g\left(\frac{x}{t}\right)\frac{dt}{t} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}F(s)G(s)ds. \tag{5.5}$$

For more details, see pages $79-83$ in [29].

The basic idea in the first method employs self-reciprocal functions. These are functions that are reproduced after integration against a kernel. The key formulas required here are the remarkable identities derived by Koshlyakov [23] for $\frac{-1}{2} < \nu < \frac{1}{2}$:

$$\int_0^\infty K_\nu(t)\left(\cos(\nu\pi)M_{2\nu}(2\sqrt{xt}) - \sin(\nu\pi)J_{2\nu}(2\sqrt{xt})\right)dt = K_\nu(x), \tag{5.6}$$

and

$$\int_0^\infty tK_\nu(t)\left(\sin(\nu\pi)J_{2\nu}(2\sqrt{xt}) - \cos(\nu\pi)L_{2\nu}(2\sqrt{xt})\right)dt = xK_\nu(x), \tag{5.7}$$

where

$$M_\nu(x) = \frac{2}{\pi}K_\nu(x) - Y_\nu(x) \text{ and } L_\nu(x) = -\frac{2}{\pi}K_\nu(x) - Y_\nu(x) \tag{5.8}$$

are the functions introduced by G. H. Hardy. It is easy to see that the above identities actually hold for $-\frac{-1}{2} < \mathrm{Re}\ \nu < \frac{1}{2}$.

Lemma 5.1. *Assume* $\pm\mathrm{Re}\,\frac{z}{2} < \mathrm{Re}\, s < \frac{3}{4}$ *and* $y > 0$. *Then*

$$\int_0^\infty x^{s-1}\left(\cos\left(\tfrac{1}{2}\pi z\right)M_z(4\pi\sqrt{xy}) - \sin\left(\tfrac{1}{2}\pi z\right)J_z(4\pi\sqrt{xy})\right)dx \tag{5.9}$$
$$= \frac{1}{2^{2s}\pi^{1+2s}y^s}\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)\left(\cos\left(\tfrac{1}{2}\pi z\right) + \cos(\pi s)\right).$$

Proof. The Mellin transform of the modified Bessel function $K_z(x)$, given in (2.8), is

$$\int_0^\infty x^{s-1}K_z(ax)dx = 2^{s-2}a^{-s}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right) \tag{5.10}$$

for $\mathrm{Re}\, s > \pm\mathrm{Re}\, z$ and $a > 0$. This yields

$$\int_0^\infty x^{s-1}\frac{2}{\pi}K_z(4\pi\sqrt{xy})dx = 2^{-2s}\pi^{-1-2s}y^{-s}\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right) \tag{5.11}$$

for Re $s > \pm$ Re $\left(\frac{z}{2}\right)$ and $y > 0$.

The Bessel function of the second kind $Y_z(x)$, satisfies [28, p. 93, formula 10.2]

$$\int_0^\infty x^{s-1} Y_z(ax)dx = -\frac{1}{\pi}2^{s-1}a^{-s}\cos\left(\tfrac{\pi}{2}(s-z)\right)\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right),$$

for $\pm$ Re $z <$ Re $s < \frac{3}{2}$, which produces

$$\int_0^\infty x^{s-1} Y_z(4\pi\sqrt{xy})dx = -\pi^{-1-2s}2^{-2s}y^{-s}\cos\left(\pi s - \tfrac{1}{2}\pi z\right)\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right),$$

for $\pm \operatorname{Re}\frac{z}{2} < \operatorname{Re} s < \frac{3}{4}$. Thus for $\pm \operatorname{Re}\frac{z}{2} < \operatorname{Re} s < \frac{3}{4}$,

$$\begin{aligned}&\int_0^\infty x^{s-1}\cos\left(\tfrac{1}{2}\pi z\right)M_z(4\pi\sqrt{xy})dx \\ &= 2^{-2s}\pi^{-1-2s}y^{-s}\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)\cos\left(\tfrac{1}{2}\pi z\right)\left(1+\cos\left(\pi s - \tfrac{1}{2}\pi z\right)\right).\end{aligned}$$

Similarly, using [28, p. 93, formula 10.1],

$$\int_0^\infty x^{s-1} J_z(ax)\,dx = \frac{1}{2}\left(\frac{a}{2}\right)^{-s}\frac{\Gamma\left(\frac{s+z}{2}\right)}{\Gamma\left(1+\frac{z-s}{2}\right)} \tag{5.12}$$

for $-\operatorname{Re} z < \operatorname{Re} s < \frac{3}{2}$, yields

$$\int_0^\infty x^{s-1} J_z(4\pi\sqrt{xy})dx = 2^{-2s}\pi^{-2s}y^{-s}\frac{\Gamma\left(s+\frac{z}{2}\right)}{\Gamma\left(1-s+\frac{z}{2}\right)}$$

for $-\operatorname{Re}\frac{z}{2} < \operatorname{Re} s < \frac{3}{4}$. Combining these evaluations completes the proof. □

Similar arguments prove the next result.

Lemma 5.2. *Assume* $\pm \operatorname{Re}\frac{z}{2} < \operatorname{Re} s < \frac{3}{4}$ *and* $y > 0$. *Then*

$$\begin{aligned}&\int_0^\infty x^{s-1}\left(\sin\left(\tfrac{1}{2}\pi z\right)J_z(4\pi\sqrt{xy}) - \cos\left(\tfrac{1}{2}\pi z\right)L_z(4\pi\sqrt{xy})\right)dx \\ &\quad = \frac{1}{2^{2s}\pi^{1+2s}y^s}\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)\left(\cos\left(\tfrac{1}{2}\pi z\right)-\cos(\pi s)\right).\end{aligned} \tag{5.13}$$

The next statement shows how to produce functions self-reciprocal with respect to the kernel (5.15).

Theorem 5.3. *Assume* $\pm \operatorname{Re}\frac{z}{2} < c = \operatorname{Re} s < 1 \pm \operatorname{Re}\frac{z}{2}$. *Define* $f(x,z)$ *by*

$$f(x,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}F(s,z)\zeta(1-s-z/2)\zeta(1-s+z/2)\,ds, \tag{5.14}$$

where $F(s,z)$ *is a function satisfying* $F(s,z) = F(1-s,z)$ *and is such that the above integral converges. Then* f *is self-reciprocal (as a function of* x*) with respect to the kernel*

$$2\pi\left(\cos\left(\tfrac{1}{2}\pi z\right)M_z(4\pi\sqrt{xy}) - \sin\left(\tfrac{1}{2}\pi z\right)J_z(4\pi\sqrt{xy})\right), \tag{5.15}$$

that is,
(5.16)
$$f(y,z) = 2\pi \int_0^\infty f(x,z)\left[\cos\left(\tfrac{1}{2}\pi z\right) M_z(4\pi\sqrt{xy}) - \sin\left(\tfrac{1}{2}\pi z\right) J_z(4\pi\sqrt{xy})\right] dx.$$

Proof. Apply Parseval's identity (5.4) to the product of the functions $f(x,z)$ and $g(x,z) = 2\pi\left[\cos\left(\frac{1}{2}\pi z\right) M_z(4\pi\sqrt{xy}) - \sin\left(\frac{1}{2}\pi z\right) J_z(4\pi\sqrt{xy})\right]$, with z a parameter. Lemma 5.1 gives the identity

$$2\pi \int_0^\infty f(x,z)\left[\cos\left(\tfrac{1}{2}\pi z\right) M_z(4\pi\sqrt{xy}) - \sin\left(\tfrac{1}{2}\pi z\right) J_z(4\pi\sqrt{xy})\right] dx = \frac{1}{2\pi i}\int_C F(s,z)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right) 2^{1-2s}\pi^{-2s}y^{-s} \times \Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)\left[\cos\left(\tfrac{1}{2}\pi z\right) + \cos(\pi s)\right] ds,$$

where C is the line $\text{Re}\, s = c$ and $\pm$ Re $\frac{z}{2} < c < \min\left\{\frac{3}{4}, 1 \pm \text{Re}\,\frac{z}{2}\right\}$. Now use the functional equation for the Riemann zeta function (2.5) in the form

$$\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\tfrac{1}{2}\pi s\right) \tag{5.17}$$

to simplify the line integral and produce

$$2\pi \int_0^\infty f(x,z)\left[\cos\left(\tfrac{1}{2}\pi z\right) M_z(4\pi\sqrt{xy}) - \sin\left(\tfrac{1}{2}\pi z\right) J_z(4\pi\sqrt{xy})\right] dx = \frac{1}{2\pi i}\int_C F(s,z)\zeta\left(1-s-\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right) y^{-s} ds.$$

This last line is $f(y,z)$ and the result has been established. □

Corollary 5.4. *Let $f(x,z)$ be as in the previous theorem. Then, if α, $\beta > 0$ and $\alpha\beta = 1$ and $-1 < \text{Re}\, z < 1$,*

$$\sqrt{\alpha}\int_0^\infty K_{z/2}(2\pi\alpha x) f(x,z) dx = \sqrt{\beta}\int_0^\infty K_{z/2}(2\pi\beta x) f(x,z) dx. \tag{5.18}$$

Proof. The identity (5.6) produces
(5.19)
$$K_{\frac{z}{2}}(2\pi\alpha x) = \frac{2\pi}{\alpha}\int_0^\infty K_{\frac{z}{2}}\left(\frac{2\pi y}{\alpha}\right)\left(\cos\left(\frac{\pi z}{2}\right) M_z(4\pi\sqrt{yx}) - \sin\left(\frac{\pi z}{2}\right) J_{2\nu}(4\pi\sqrt{yx})\right) dy.$$

Thus,

$$\int_0^\infty K_{\frac{z}{2}}(2\pi\alpha x) f(x,z)\, dx = \frac{2\pi}{\alpha}\int_0^\infty K_{\frac{z}{2}}\left(\frac{2\pi y}{\alpha}\right)\int_0^\infty f(x,z)\left(\cos\left(\frac{\pi z}{2}\right) M_z(4\pi\sqrt{yx}) - \sin\left(\frac{\pi z}{2}\right) J_{2\nu}(4\pi\sqrt{yx})\right) dx\, dy,$$

where the interchange of the order of integration can be easily justified. Now apply (5.16) and use the fact $\alpha\beta = 1$, to deduce (5.18). □

The next statement admits a similar proof as the previous theorem.

Theorem 5.5. *Assume* $\pm \operatorname{Re} \frac{z}{2} < c = \operatorname{Re} s < 1 \pm \operatorname{Re} \frac{z}{2}$. *Define* $f(x,z)$ *by*

$$f(x,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{F(s,z)}{(2\pi)^{2s}}\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)x^{-s}\,ds, \tag{5.20}$$

where $F(s,z)$ *is a function satisfying* $F(s,z) = F(1-s,z)$ *and is such that the above integral converges. Then* f *is self-reciprocal (as a function of* x*) with respect to the kernel*

$$2\pi\left(\sin\left(\tfrac{1}{2}\pi z\right)J_z(4\pi\sqrt{xy}) - \cos\left(\tfrac{1}{2}\pi z\right)L_z(4\pi\sqrt{xy})\right), \tag{5.21}$$

that is,

$$f(y,z) = 2\pi\int_0^\infty f(x,z)\left[\sin\left(\tfrac{1}{2}\pi z\right)J_z(4\pi\sqrt{xy}) - \cos\left(\tfrac{1}{2}\pi z\right)L_z(4\pi\sqrt{xy})\right]dx. \tag{5.22}$$

Corollary 5.6. *Let* $f(x,z)$ *be as in the previous theorem. Then, if* α, $\beta > 0$ *and* $\alpha\beta = 1$ *and* $-1 < \operatorname{Re} z < 1$,

$$\sqrt{\alpha^3}\int_0^\infty xK_{z/2}(2\pi\alpha x)f(x,z)dx = \sqrt{\beta^3}\int_0^\infty xK_{z/2}(2\pi\beta x)f(x,z)dx. \tag{5.23}$$

Proof. The proof of similar to that of Corollary 5.4, so details are omitted. □

The proofs of the theorems stated in Section 4 are given now. In the proofs below, the convergence of the integrals involving the gamma function can be easily proved using Stirling's formula for $\Gamma(s)$, $s = \sigma + it$, in a vertical strip $\alpha \leq \sigma \leq \beta$ given by

$$|\Gamma(\sigma+it)| = (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|}\left(1+O\left(\frac{1}{|t|}\right)\right) \tag{5.24}$$

as $|t| \to \infty$. The vanishing of the integrals which involve gamma function along the horizontal segments of a contour as the height $T \to \infty$ is also established using (5.24).

The interchange of the order of summation and integration, or of the order of integration in the case of double integrals, is permissible because of absolute convergence of the series and integrals involved.

Proof of Theorem 4.6. Take $F(s,z) \equiv 1$ in Theorem 5.5. The conclusion requires the evaluation of

$$\begin{aligned} f(x,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} &\left(4\pi^2 x\right)^{-s} \\ &\times\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)ds \end{aligned} \tag{5.25}$$

for $\pm \operatorname{Re} \frac{z}{2} < c = \operatorname{Re} s < 1 \pm \operatorname{Re} \frac{z}{2}$. It will be shown next that

$$f(x,z) = \Phi(x,z), \tag{5.26}$$

using the notation of Theorem 4.6.

This computation begins by using the expansion

$$\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)=\sum_{n=1}^{\infty}\frac{\sigma_{-z}(n)}{n^{s-z/2}} \tag{5.27}$$

valid for $\text{Re}\, s > 1 \pm \text{Re}\,\frac{z}{2}$. See [33, p. 8, equation 1.3.1]. Thus, in order to use (5.27) in (5.25), it is required to shift the line of integration to the vertical line $\lambda = \text{Re}\, s > 1 \pm \text{Re}\,\frac{z}{2}$. This shift captures two poles of the integrand at $s = 1 + z/2$ and $s = 1 - z/2$. Since the contributions of the horizontal segments at $\text{Im}\, s = \pm T$ vanish as $T \to \infty$, the residue theorem gives

$$\begin{aligned} f(x,z) &= \sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)(4\pi^2nx)^{-s}ds \\ &- \lim_{s\to 1+z/2}\left(s-\frac{z}{2}-1\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)(4\pi^2x)^{-s} \\ &- \lim_{s\to 1-z/2}\left(s+\frac{z}{2}-1\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)(4\pi^2x)^{-s}. \end{aligned}$$

The line integral above is evaluated using (5.10) that gives

$$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)(4\pi^2nx)^{-s}ds = 2K_z(4\pi\sqrt{nx}). \tag{5.28}$$

Then the residues at the poles are computed using

$$\lim_{s\to 1}(s-1)\zeta(s)=1, \tag{5.29}$$

that gives

$$\begin{aligned} f(x,z) &= 2\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_z(4\pi\sqrt{nx}) \\ &\qquad - \frac{\Gamma(1+z)\zeta(1+z)x^{-1-z/2}}{(2\pi)^{2+z}} - \frac{\Gamma(1-z)\zeta(1-z)x^{-1+z/2}}{(2\pi)^{2-z}}. \end{aligned}$$

This shows $f(x,z) = \Phi(x,z)$. Corollary (5.6) now establishes the first equality in (4.12).

The equality between the extreme left and right sides of (4.12) is establihsed next. The invariance of the latter under $\alpha \to 1/\alpha$ then easily establishes the other equality, thus giving another proof of the transformation.

The proof begins by replacing s by $s+1$, a by $2\pi\alpha$ and z by $z/2$ in (5.10) to obtain

$$\begin{aligned} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2^{s-1}(2\pi\alpha)^{-s-1}\Gamma\left(\frac{s+1}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s+1}{2}+\frac{z}{4}\right)x^{-s}ds \\ = xK_{z/2}(2\pi\alpha x) \end{aligned} \tag{5.30}$$

for Re $s > -1\pm$ Re $\frac{z}{2}$. Then (5.4), (5.25) and (5.30) give

$$\sqrt{\alpha^3}\int_0^\infty xK_{z/2}(2\pi\alpha x)\Phi(x,z)dx = \frac{\sqrt{\alpha^3}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{(\pi\alpha)^{s-2}}{4}\Gamma\left(1-\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(1-\frac{s}{2}+\frac{z}{4}\right)\right] \times\frac{1}{(2\pi)^{2s}}\Gamma\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)ds, \tag{5.31}$$

for $\pm\,\mathrm{Re}\,\frac{z}{2} < c = \mathrm{Re}\,s < 1\pm\mathrm{Re}\,\frac{z}{2}$. Now use the relation

$$\Gamma(s)\Gamma\left(s+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2s-1}}\Gamma(2s) \tag{5.32}$$

to express each of $\Gamma(s\pm z/2)$ as a product of two gamma factors and obtain

$$\sqrt{\alpha^3}\int_0^\infty xK_{z/2}(2\pi\alpha x)\Phi(x,z)dx = \frac{1}{32\pi^4 i\sqrt{\alpha}}\int_{c-i\infty}^{c+i\infty}\Gamma\left(1-\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(1-\frac{s}{2}+\frac{z}{4}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right) \times\Gamma\left(\frac{s}{2}-\frac{z}{4}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}+\frac{1}{2}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{\alpha}\right)^{-s}ds. \tag{5.33}$$

The integrand is now expressed in terms of the Riemann ξ-function (1.5) with $c=\frac{1}{2}$ and $-1 < \mathrm{Re}\,z < 1$. This yields

$$\sqrt{\alpha^3}\int_0^\infty xK_{z/2}(2\pi\alpha x)\Phi(x,z)dx = \frac{1}{128\pi^4 i\sqrt{\alpha}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(-\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(-\frac{s}{2}+\frac{z}{4}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}-\frac{1}{2}\right) \times\Gamma\left(\frac{s}{2}+\frac{z}{4}-\frac{1}{2}\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)\alpha^s ds. \tag{5.34}$$

The last step is to use the identity [2]

$$\int_0^\infty f\left(z,\frac{t}{2}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\cos\left(\tfrac{1}{2}t\log\alpha\right)dt = \frac{1}{i\sqrt{\alpha}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\left(z,s-\tfrac{1}{2}\right)\phi\left(z,\tfrac{1}{2}-s\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)\alpha^s ds, \tag{5.35}$$

established in [8], with $f(z,t)=\phi(z,it)\phi(z,-it)$, to rewrite the integral on the right side of (5.34). Taking

$$\phi(z,s) = \frac{1}{8\sqrt{2}\,\pi^2}\Gamma\left(-\frac{s}{2}+\frac{z}{4}-\frac{1}{4}\right)\Gamma\left(-\frac{s}{2}-\frac{z}{4}-\frac{1}{4}\right) \tag{5.36}$$

[2]There is a typo in the identity in [8]. One of the Ξ-functions should have $(t-iz)/2$ as its argument.

produces

$$\sqrt{\alpha^3}\int_0^\infty xK_{z/2}(2\pi\alpha x)\Phi(x,z)dx = \frac{1}{128\pi^4}\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Gamma\left(\frac{-z-1+it}{4}\right) \times\Gamma\left(\frac{-z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\cos\left(\tfrac{1}{2}t\log\alpha\right)dt. \tag{5.37}$$

Finally, the integrand on the right side can be simplified, using $\Gamma(u+1) = u\Gamma(u)$, to the form given in (4.12). This completes the proof.

Proof of Theorem 4.8. In view of Corollary 4.7, it suffices to show that

$$\sqrt{\alpha^3}\int_0^\infty xK_0(2\pi\alpha x)\left(2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{nx}) + \frac{\log(4\pi^2 x)}{4\pi^2 x}\right)dx = \frac{\sqrt{\alpha}}{16\pi^2}\int_0^\infty\int_0^\infty \frac{y}{(y^2+t^2)^{3/2}}\left(J_0(2\alpha y) + \frac{4\pi t}{e^{2\pi t}-1}\left(\frac{1}{e^{2\pi\alpha y}-1} - \frac{1}{2\pi\alpha y}\right)\right)dy\,dt. \tag{5.38}$$

The proof begins with an auxiliary result.

Lemma 5.7. *For $\alpha, t > 0$, we have*

$$\int_0^\infty \frac{xK_0(2\pi\alpha x)}{e^{2\pi x/t}-1}dx = \frac{t}{8\pi\alpha} - \frac{t^2}{2}\int_0^\infty xJ_0(2\pi t\alpha x)\left(\psi(x+1)-\log x\right)dx. \tag{5.39}$$

Proof. Let $z = 0$ in (5.30) to see that for $\lambda = \text{Re } s > -1$,

$$xK_0(2\pi\alpha x) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} 2^{s-1}(2\pi\alpha)^{-s-1}\Gamma^2\left(\frac{s+1}{2}\right)x^{-s}\,ds. \tag{5.40}$$

It is well-known that for Re $s > 1$,

$$\int_0^\infty \frac{y^{s-1}}{e^y-1}dy = \Gamma(s)\zeta(s). \tag{5.41}$$

Let $y = 2\pi x/t$ with $t > 0$ to obtain

$$\frac{1}{2\pi i}\int_{\lambda'-i\infty}^{\lambda'+i\infty}\Gamma(s)\zeta(s)\left(\frac{2\pi x}{t}\right)^{-s}ds = \frac{1}{e^{2\pi x/t}-1}, \tag{5.42}$$

for $\lambda' = \text{Re}\, s > 1$. Using (5.40), (5.42) and Parseval's identity (5.4), for $-1 < c = \text{Re } s < 0$, give

$$\int_0^\infty \frac{xK_0(2\pi\alpha x)}{e^{2\pi x/t}-1}dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{2\pi}\right)^{1-s}\Gamma(1-s)\zeta(1-s)2^{s-1}(2\pi\alpha)^{-s-1}\Gamma^2\left(\frac{s+1}{2}\right)ds.$$

Now use the functional equation (5.17) and a variant of the reflection formula for the gamma function, namely,

$$\Gamma\left(\frac{1}{2}+w\right)\Gamma\left(\frac{1}{2}-w\right)=\frac{\pi}{\cos\pi w},\quad w-\tfrac{1}{2}\notin\mathbb{Z}, \tag{5.43}$$

in the above equation and simplify to deduce that

$$\int_0^\infty \frac{xK_0(2\pi\alpha x)}{e^{2\pi x/t}-1}\,dx=\frac{t}{8\pi i\alpha}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\frac{\zeta(s)}{\sin\pi s}(t\pi\alpha)^{-s}\,ds. \tag{5.44}$$

Next, shift the line of integration from Re $s=c$, $-1<c<0$, to Re $s=c'$, $0<c'<1/2$ and apply the residue theorem by considering a rectangular contour. In doing so, one needs to consider the contribution from the pole of order 1 at $s=0$. Noting that the integrals along the horizontal segments of the contour tend to zero as the height tends to ∞, gives

(5.45)

$$\begin{aligned}&\int_0^\infty \frac{xK_0(2\pi\alpha x)}{e^{2\pi x/t}-1}\,dx\\&=\frac{t}{4\alpha}\left\{\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\frac{\zeta(s)}{\sin\pi s}(t\pi\alpha)^{-s}\,ds-\lim_{s\to 0}s\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\frac{\zeta(s)}{\sin\pi s}(t\pi\alpha)^{-s}\right\}\\&=\frac{t}{4\alpha}\left(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\frac{\zeta(s)}{\sin\pi s}(t\pi\alpha)^{-s}\,ds+\frac{1}{2\pi}\right).\end{aligned}$$

To evaluate the above integral, first use (5.12) with $z=0, a=2$ and s replaced by $s+1$ so that for $-1<d=\text{Re }s<\frac{1}{2}$,

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}x^{-s}\,ds=2xJ_0(2x). \tag{5.46}$$

The next steps employs a formula of Kloosterman [33, p. 24-25, equations (2.9.1), (2.9.2)], for $0<d'=\text{Re }s<1$,

$$\frac{\zeta(s)}{\sin\pi s}=-\frac{1}{\pi}\int_0^\infty(\psi(x+1)-\log x)\,x^{-s}\,dx. \tag{5.47}$$

Replacing x by $1/x$ in the above formula, produces

$$\frac{1}{2\pi i}\int_{d'-i\infty}^{d'+i\infty}\frac{\zeta(s)}{\sin\pi s}x^{-s}\,ds=-\frac{1}{\pi x}\left(\psi\left(\frac{1}{x}+1\right)+\log x\right), \tag{5.48}$$

for $0 < d' = \text{Re } s < 1$. Since (5.46) and (5.48) are both valid in the region $0 < \text{Re } s < \frac{1}{2}$, using (5.5), it follows that

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\frac{\zeta(s)}{\sin\pi s}(t\pi\alpha)^{-s}\,ds$$
$$= \frac{-1}{\pi^2 t\alpha}\int_0^\infty 2xJ_0(2x)\left(\psi\left(\frac{x}{\pi t\alpha}+1\right)-\log\frac{x}{\pi t\alpha}\right)dx. \tag{5.49}$$

Now let $x \to \pi t\alpha x$ and substitute in (5.45) to obtain (5.39). □

A proof of (5.38) is presented next.

Proof. Let
(5.50)
$$H(\alpha) := \sqrt{\alpha^3}\int_0^\infty xK_0(2\pi\alpha x)\left(2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{nx})+\frac{\log(4\pi^2x)}{4\pi^2x}\right)dx.$$

Page 254 in the Lost Notebook [32] (see also equation (4.1) in [3]) gives

$$\int_0^\infty \frac{dt}{t(e^{2\pi t}-1)(e^{2\pi x/t}-1)} = 2\sum_{n=1}^\infty d(n)K_0\left(4\pi\sqrt{nx}\right). \tag{5.51}$$

Hence

$$H(\alpha) = \sqrt{\alpha^3}\Bigg\{\frac{\log(4\pi^2)}{4\pi^2}\int_0^\infty K_0(2\pi\alpha x)\,dx+\frac{1}{4\pi^2}\int_0^\infty K_0(2\pi\alpha x)\log x\,dx$$
$$+\int_0^\infty xK_0(2\pi\alpha x)\,dx\int_0^\infty\frac{dt}{t(e^{2\pi t}-1)(e^{2\pi x/t}-1)}\Bigg\}. \tag{5.52}$$

Now formula 6.511.12 in [12] gives

$$\int_0^\infty K_0(2\pi\alpha x)\,dx = \frac{1}{4\alpha}, \tag{5.53}$$

and formula 2.16.20.1 on page 365 of [30] states that for $|\text{Re}\,w| > \text{Re}\,\nu$ and real $m > 0$,

$$\int_0^\infty x^{w-1}K_\nu(mx)\log x\,dx = \frac{2^{w-3}}{m^w}\Gamma\left(\frac{w+\nu}{2}\right)\Gamma\left(\frac{w-\nu}{2}\right)$$
$$\left\{\psi\left(\frac{w+\nu}{2}\right)+\psi\left(\frac{w-\nu}{2}\right)-2\log\left(\frac{m}{2}\right)\right\}. \tag{5.54}$$

Then (5.53) and the above formula with $w = 1$, $\nu = 0$ and $m = 2\pi\alpha$ converts (5.52) to

$$H(\alpha) = \sqrt{\alpha^3}\left\{\frac{\log(4\pi^2)}{16\pi^2\alpha}-\frac{1}{4\pi^2}\left(\frac{\gamma+\log(4\pi\alpha)}{4\alpha}\right)+\int_0^\infty\frac{dt}{t(e^{2\pi t}-1)}\int_0^\infty\frac{xK_0(2\pi\alpha x)}{e^{2\pi x/t}-1}\,dx\right\}$$
(5.55)
$$= \frac{\sqrt{\alpha}}{16\pi^2}\left\{-\left(\gamma-\log\left(\frac{\pi}{\alpha}\right)\right)+16\pi^2\alpha\int_0^\infty\frac{dt}{t(e^{2\pi t}-1)}\int_0^\infty\frac{xK_0(2\pi\alpha x)}{e^{2\pi x/t}-1}\,dx\right\}.$$

Lemma 5.7 along with the integral representation [2, equation 3.5]

$$\gamma-\log\left(\frac{\pi}{\alpha}\right)=\int_0^\infty\left(\frac{2\pi}{e^{2\pi t}-1}-\frac{e^{-2\alpha t}}{t}\right)dt \tag{5.56}$$

is used in (5.55) to obtain

$$\begin{aligned}H(\alpha)&=\frac{\sqrt{\alpha}}{16\pi^2}\Bigg\{\int_0^\infty\left(\frac{e^{-2\alpha t}}{t}-\frac{2\pi}{e^{2\pi t}-1}\right)dt\\&\qquad+\int_0^\infty\frac{dt}{t(e^{2\pi t}-1)}\left(2\pi t-8\pi^2\alpha t^2\int_0^\infty xJ_0(2\pi t\alpha x)\left(\psi(x+1)-\log x\right)dx\right)\Bigg\}\\&=\frac{\sqrt{\alpha}}{16\pi^2}\int_0^\infty\left(\frac{e^{-2\alpha t}}{t}-\frac{8\pi^2\alpha t}{e^{2\pi t}-1}\int_0^\infty xJ_0(2\pi t\alpha x)\left(\psi(x+1)-\log x\right)dx\right)dt.\end{aligned} \tag{5.57}$$

The standard formulas

$$2\pi\int_0^\infty\left(\frac{1}{e^{2\pi y}-1}-\frac{1}{2\pi y}\right)e^{-2\pi xy}\,dy=\log x-\psi(x+1), \tag{5.58}$$

which can be obtained from [12, p. 360, 3.427.7] and [12, p. 702, 6.623.2]

$$\int_0^\infty e^{-ax}J_\nu(bx)x^{\nu+1}\,dx=\frac{2a(2b)^\nu\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}(a^2+b^2)^{\nu+\frac{3}{2}}}, \tag{5.59}$$

for $\mathrm{Re}\,\nu>-1$ and $\mathrm{Re}\,a>|\,\mathrm{Im}\,b|$. Substitute (5.58) on the extreme right of (5.57), and then use (5.59) with $\nu=0, a=2\pi y$ and $b=2\pi t\alpha$ in the resulting equation to see (after simplification) that

$$\begin{aligned}H(\alpha)&=\frac{\sqrt{\alpha}}{16\pi^2}\int_0^\infty\left(\frac{e^{-2\alpha t}}{t}+\frac{4\pi\alpha t}{e^{2\pi t}-1}\int_0^\infty\frac{y}{(y^2+(\alpha t)^2)^{3/2}}\left(\frac{1}{e^{2\pi y}-1}-\frac{1}{2\pi y}\right)dy\right)dt\\&=\frac{\sqrt{\alpha}}{16\pi^2}\int_0^\infty\left(\frac{e^{-2\alpha t}}{t}+\frac{4\pi t}{e^{2\pi t}-1}\int_0^\infty\frac{y}{(y^2+t^2)^{3/2}}\left(\frac{1}{e^{2\pi\alpha y}-1}-\frac{1}{2\pi\alpha y}\right)dy\right)dt\end{aligned} \tag{5.60}$$

Finally, use [12, p. 675, 6.554.4]

$$\int_0^\infty\frac{yJ_0(2\alpha y)}{(y^2+t^2)^{3/2}}\,dy=\frac{e^{-2\alpha t}}{t},\qquad\text{for }\alpha,\,t>0, \tag{5.61}$$

on the extreme right of (5.60) to prove (5.38). □

Proof of Theorem 4.5. The first proof uses Theorem 5.3 with $F(s,z)=\Gamma\left(s+z/2\right)\Gamma(1-s+z/2)$. This requires the evaluation of

$$\begin{aligned}f(x,z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}&\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\\&\times\zeta\left(1-s-\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)x^{-s}ds\end{aligned} \tag{5.62}$$

for $\pm\,\mathrm{Re}\,\frac{z}{2}<c=\mathrm{Re}\,s<1\pm\mathrm{Re}\,\frac{z}{2}$.

First assume $c = \text{Re } s > 1 \pm \text{ Re } \frac{z}{2}$. Denote the right-hand side of (5.62) by $I(x,z)$. The change of variables $s = 1-w$, gives

$$I(x,w) = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} H(x,w,z)dw, \tag{5.63}$$

where $\lambda = \text{Re}\, w < \pm\, \text{Re}\, \frac{z}{2}$ and

$$H(x,w,z) = \Gamma\left(w+\frac{z}{2}\right)\Gamma\left(1-w+\frac{z}{2}\right)\zeta\left(w-\frac{z}{2}\right)\zeta\left(w+\frac{z}{2}\right)x^{w-1}. \tag{5.64}$$

The line of integration is now shifted from $\text{Re}\, w = \lambda < \pm\, \text{Re}\, \frac{z}{2}$ to $\text{Re}\, w = \lambda' > 1 \pm \text{Re}\, \frac{z}{2}$. These two lines are closed to form the rectangular contour with sides $(\lambda - iT, \lambda' - iT)$, $(\lambda' - iT, \lambda' + iT)$, $(\lambda' + iT, \lambda + iT)$, $(\lambda + iT, \lambda - iT)$, where $T > 0$. This shift encounters poles at $w = -z/2$, $1 + z/2$ and $1 - z/2$ of orders 1, 2 and 1, respectively. The residue theorem gives

$$\begin{aligned}\int_{\lambda - iT}^{\lambda + iT} H(x,w,z)dw = &\left[\int_{\lambda - iT}^{\lambda' - iT} + \int_{\lambda' - iT}^{\lambda' + iT} + \int_{\lambda' + iT}^{\lambda + iT}\right] H(x,w,z)dw \\ &- 2\pi i\left[R(-z/2) + R(1+z/2) + R(1-z/2)\right]\end{aligned} \tag{5.65}$$

where $R(a)$ denotes the residue at the pole a. It is easy to see that the integrals along the horizontal segments tend to 0 as $T \to \infty$. Therefore

$$\begin{aligned}\int_{\lambda - i\infty}^{\lambda + i\infty} H(x,w,z)dw = &\int_{\lambda' - i\infty}^{\lambda' + i\infty} H(x,w,z)dw \\ &- 2\pi i\left[R(-z/2) + R(1+z/2) + R(1-z/2)\right].\end{aligned} \tag{5.66}$$

The line integral is evaluated using (5.27) to obtain

$$\begin{aligned}&\int_{\lambda' - i\infty}^{\lambda' + i\infty} H(x,w,z)dw = \\ &\quad \frac{1}{x}\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}\int_{\lambda' - i\infty}^{\lambda' + i\infty}\Gamma\left(w+\frac{z}{2}\right)\Gamma\left(1-w+\frac{z}{2}\right)\left(\frac{n}{x}\right)^{-w}dw,\end{aligned} \tag{5.67}$$

for $\text{Re}\,(w \pm z/2) > 1$. Now, for $0 < w_0 = \text{Re}\, w < \text{Re}\, z$,

$$\frac{1}{2\pi i}\int_{w_0 - i\infty}^{w_0 + i\infty}\frac{\Gamma(w)\Gamma(z-w)}{\Gamma(z)}x^{-w}dw = \frac{1}{(1+x)^z}, \tag{5.68}$$

which, upon replacement of w by $w + \frac{z}{2}$ and z by $1+z$, gives

$$\frac{1}{2\pi i}\int_{d - i\infty}^{d + i\infty}\Gamma\left(w+\frac{z}{2}\right)\Gamma\left(1-w+\frac{z}{2}\right)x^{-w}\,dw = \frac{x^{\frac{z}{2}}\Gamma(1+z)}{(1+x)^{1+z}} \tag{5.69}$$

for $-\operatorname{Re}\frac{z}{2} < d = \operatorname{Re} w < 1 + \operatorname{Re}\frac{z}{2}$. Another application of the residue theorem leads to

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \Gamma\left(w+\frac{z}{2}\right)\Gamma\left(1-w+\frac{z}{2}\right)x^{-w}dw = \Gamma(1+z)\left[\frac{x^{z/2}}{(1+x)^{1+z}} - x^{-1-z/2}\right] \tag{5.70}$$

for $d = \operatorname{Re} w > 1 + \operatorname{Re}\frac{z}{2}$. Then (5.67) and (5.70) give

(5.71)

$$\frac{1}{2\pi i}\int_{\lambda'-i\infty}^{\lambda'+i\infty} H(x,w,z)dw = x^{z/2}\Gamma(1+z)\sum_{n=1}^{\infty}\sigma_{-z}(n)\left(\frac{n^z}{(n+x)^{z+1}} - \frac{1}{n}\right),$$

and (5.66) gives

(5.72)

$$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} H(x,w,z)dw = x^{z/2}\Gamma(1+z)\sum_{n=1}^{\infty}\sigma_{-z}(n)\left(\frac{n^z}{(n+x)^{z+1}} - \frac{1}{n}\right) - R(-z/2) - R(1+z/2) - R(1-z/2).$$

The computation of the residues at the poles yields the values

(5.73)

$$\begin{aligned} R(-z/2) &= -\tfrac{1}{2}\Gamma(1+z)\zeta(-z)x^{-1-z/2} \\ R(1+z/2) &= -x^{z/2}\Gamma(1+z)\left[(2\gamma+\log x+\psi(1+z))\zeta(1+z)+\zeta'(1+z)\right] \\ R(1-z/2) &= x^{-z/2}\Gamma(z)\zeta(1-z). \end{aligned}$$

For example,

$$\begin{aligned} R(-z/2) &= \lim_{w\to-\frac{z}{2}} H(x,w,z) \\ &= \lim_{w\to-\frac{z}{2}} \left(w+\frac{z}{2}\right)\Gamma\left(w+\frac{z}{2}\right)\Gamma\left(1-w+\frac{z}{2}\right)\zeta\left(w-\frac{z}{2}\right)\zeta\left(w+\frac{z}{2}\right)x^{w-1} \\ &= \lim_{w\to-\frac{z}{2}} \Gamma\left(w+\frac{z}{2}+1\right)\Gamma\left(1-w+\frac{z}{2}\right)\zeta\left(w-\frac{z}{2}\right)\zeta\left(w+\frac{z}{2}\right)x^{w-1} \\ &= -\frac{1}{2}\Gamma(1+z)\zeta(-z)x^{-z/2-1}, \end{aligned}$$

using $\Gamma(1) = 1$ and $\zeta(0) = -1/2$.

Now (4.8), (5.63), (5.72) and (5.73) give for $c = \operatorname{Re} s > 1 \pm \operatorname{Re}\frac{z}{2}$,

$$I(x,z) = \Lambda(x,z) + \frac{1}{2}\Gamma(1+z)\zeta(-z)x^{-1-\frac{z}{2}}. \tag{5.74}$$

Finally, if $\pm\operatorname{Re}\frac{z}{2} < c = \operatorname{Re} s < 1 \pm \operatorname{Re}\frac{z}{2}$, the residue theorem again produces

$$I(x,z) = \Lambda(x,z). \tag{5.75}$$

Then, (5.62) implies $f(x,z) = \Lambda(x,z)$. Corollary 5.4 now establishes the first equality in (4.9).

Second proof: An alternative proof of the first equality begins with the integral on the right-hand side of (4.9) written in the form

$$I(z;\alpha) = \int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Gamma\left(\frac{z+1+it}{4}\right) \times\Gamma\left(\frac{z+1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos(\frac{1}{2}t\log\alpha)}{(z+1)^2+t^2}\,dt. \tag{5.76}$$

Let

$$f(z,t) = \frac{1}{(z+1)^2+4t^2}\Gamma\left(\frac{z-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{z-1}{4}-\frac{it}{2}\right)\Gamma\left(\frac{z+1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{z+1}{4}-\frac{it}{2}\right) \tag{5.77}$$

that admits a factorization of the type (1.9) with

$$\phi(z,s) = \frac{1}{1+z+2s}\Gamma\left(\frac{z-1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{z+1}{4}+\frac{s}{2}\right). \tag{5.78}$$

Using (5.35),

$$\begin{aligned} I(z,\alpha) &= \frac{1}{i\sqrt{\alpha}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{1}{(z+2s)(z+2-2s)} \\ &\times\Gamma\left(\frac{z}{4}+\frac{s-1}{2}\right)\Gamma\left(\frac{z}{4}+\frac{s}{2}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{z}{4}+\frac{1-s}{2}\right) \\ &\times\frac{1}{2}\left(s-\frac{z}{2}\right)\left(s-\frac{z}{2}-1\right)\pi^{-(s-z/2)/2}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right) \\ &\times\frac{1}{2}\left(s+\frac{z}{2}\right)\left(s+\frac{z}{2}-1\right)\pi^{-(s+z/2)/2}\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s+\frac{z}{2}\right)\alpha^s\,ds. \end{aligned}$$

Simplifying the integrand, this is written as

$$I(z,\alpha) = \frac{\pi}{2^{z+1}i\sqrt{\alpha}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} G(\alpha,s,z)ds, \tag{5.79}$$

with

$$G(\alpha,s,z) = \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(\frac{z}{2}-s+1\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right) \times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{\alpha}\right)^{-s}. \tag{5.80}$$

To evaluate this last integral one employs (5.27). As before, this requires to move the line of integration from $\operatorname{Re} s = \frac{1}{2}$ to $\operatorname{Re} s = \frac{3}{2}$. In this shift, one encounters a simple pole at $s = 1 - z/2$ and a double pole at $s = 1 + z/2$.

The residue theorem produces

(5.81)

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} G(\alpha,s,z)ds = \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} G(\alpha,s,z)ds - 2\pi i\left(R(1-z/2)+R(1+z/2)\right),$$

where $R(a)$ is the residue of $G(\alpha,s,z)$ at the pole $s=a$. A direct computation shows that

$$R(1-z/2) = \frac{1}{z}\alpha^{1-z/2}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\Gamma(z+1)\zeta(z) \tag{5.82}$$

and

(5.83)

$$\begin{aligned}R(1+z/2) = &-\frac{\alpha^{1+z/2}}{2\pi^{\frac{1}{2}(z+1)}}\Gamma\left(\frac{z+1}{2}\right)\Gamma(z+1)\\ &\times\left[\left(3\gamma-2\log(2\pi/\alpha)+\psi\left(\tfrac{1}{2}(z+1)\right)+2\psi(z+1)\right)\zeta(z+1)+2\zeta'(z+1)\right].\end{aligned}$$

The expansion (5.27) gives

$$\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} G(\alpha,s,z)ds =$$

$$\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(\frac{z}{2}-s+1\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\left(\frac{\pi n}{\alpha}\right)^{-s}ds.$$

In order to evaluate the integral on the right-hand side, one would like to use (5.5) with

(5.84)

$$F(s)=\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(\frac{z}{2}-s+1\right) \text{ and } G(s)=\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right).$$

Now (5.10) yields

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)x^{-s}ds = 4K_{z/2}(2x), \tag{5.85}$$

for $c=\operatorname{Re}s>\pm\operatorname{Re}z/2$ and this is true when $c=\frac{3}{2}$. However, (5.69) (with w replaced by s) holds only for $-\operatorname{Re}\frac{z}{2}<d=\operatorname{Re}s<1+\operatorname{Re}\frac{z}{2}$, which is not satisfied when $\operatorname{Re}s=\frac{3}{2}$ and $-1<\operatorname{Re}z<1$. Thus, the line of integration has to be moved from $\operatorname{Re}s=\frac{3}{2}$ to $\operatorname{Re}s=\frac{1}{2}$. This process captures a pole at $s = 1 + \frac{z}{2}$ and the residue theorem yields

$$\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(\frac{z}{2}-s+1\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\left(\frac{\pi n}{\alpha}\right)^{-s} ds$$
$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(\frac{z}{2}-s+1\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\left(\frac{\pi n}{\alpha}\right)^{-s} ds$$
$$- 2\pi i\Gamma(z+1)\Gamma\left(\frac{z+1}{2}\right)\frac{\alpha^{1+z/2}}{\pi^{\frac{z+1}{2}} n^{1+\frac{z}{2}}}.$$

Now (5.5), (5.69), (5.84) and (5.85) yield

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(\frac{z}{2}-s+1\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\left(\frac{\pi n}{\alpha}\right)^{-s} ds$$
$$= 4\Gamma(z+1)\int_0^\infty \frac{x^{z/2}K_{z/2}\left(\frac{2\pi n x}{\alpha}\right)}{(1+x)^{z+1}}\, dx \tag{5.86}$$

and so this leads to

$$\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} G(\alpha,s,z)ds =$$
$$8\pi i\Gamma(z+1)\int_0^\infty x^{z/2}K_{z/2}\left(\frac{2\pi x}{\alpha}\right)\sum_{n=1}^\infty \sigma_{-z}(n)\left(\frac{n^z}{(n+x)^{z+1}}-\frac{1}{n}\right), \tag{5.87}$$

where we used the integral representation

$$\Gamma\left(\frac{z+1}{2}\right)\frac{\alpha^{1+\frac{z}{2}}}{\pi^{\frac{z+1}{2}} n^{1+\frac{z}{2}}} = 4\int_0^\infty x^{z/2}K_{\frac{z}{2}}\left(\frac{2\pi n x}{\alpha}\right)dx$$

has been used. Using (5.81), the integral (5.79) now becomes

$$I(z,\alpha) = \frac{\pi^2}{2^z\sqrt{\alpha}}\left\{-\left(R\left(1-\frac{z}{2}\right)+R\left(1+\frac{z}{2}\right)\right)\right.$$
$$\left. +4\Gamma(z+1)\int_0^\infty x^{z/2}K_{z/2}\left(\frac{2\pi x}{\alpha}\right)\sum_{n=1}^\infty \sigma_{-z}(n)\left(\frac{n^z}{(n+x)^{z+1}}-\frac{1}{n}\right)\right\}. \tag{5.88}$$

The next step in the proof is to obtain an integral representation for the sum of the residues in the form

$$\int_0^\infty x^{z/2}K_{z/2}(2\pi\alpha x)\lambda(x,z)dx$$

for an appropriate function $\lambda(x,z)$.

The choice $w = 1 + z/2$, $\nu = z/2$ and $m = 2\pi/\alpha$ in (5.54) gives

$$\int_0^\infty 4x^{z/2} K_{z/2}\left(\frac{2\pi x}{\alpha}\right) \log x \, dx = \frac{\alpha^{1+z/2}}{2\pi^{\frac{z+1}{2}}}\Gamma\left(\frac{z+1}{2}\right)\left[-\gamma - 2\log\left(\frac{2\pi}{\alpha}\right) + \psi\left(\frac{z+1}{2}\right)\right]. \tag{5.89}$$

Now (5.10) produces

$$\int_0^\infty 4x^{z/2} K_{z/2}\left(\frac{2\pi x}{\alpha}\right) \frac{x^{-z}}{-z}\zeta(1-z)dx = -\frac{1}{z}\pi^{-z/2}\alpha^{1-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z), \tag{5.90}$$

and also

$$\begin{aligned}\int_0^\infty & 4x^{z/2} K_{z/2}\left(\frac{2\pi x}{\alpha}\right)\left[\left(2\gamma + \psi(z+1)\right)\zeta(z+1) + \zeta'(z+1)\right] dx \\ &= \frac{\alpha^{1+z/2}}{\pi^{\frac{z+1}{2}}}\Gamma\left(\frac{z+1}{2}\right)\left[\left(2\gamma + \psi(z+1)\right)\zeta(z+1) + \zeta'(z+1)\right].\end{aligned} \tag{5.91}$$

Now (5.82), (5.83), (5.89), (5.90) and (5.91) yield

$$\begin{aligned} & R(1-\frac{z}{2}) + R(1+\frac{z}{2}) \\ &= -\Gamma(z+1)\int_0^\infty 4x^{\frac{z}{2}} K_{\frac{z}{2}}\left(\frac{2\pi x}{\alpha}\right)\left\{\frac{x^{-z}}{-z}\zeta(1-z) + (2\gamma + \log x + \psi(z+1))\zeta(z+1) + \zeta'(z+1)\right\} dx.\end{aligned} \tag{5.92}$$

Substituting (5.92) in (5.88) gives the equality between the extreme left and right sides of (4.9), and the invariance of the right side under $\alpha \to 1/\alpha$ establishes the other as well.

Proof of Theorem 4.9. Since the proof of Theorem 4.9 is similar to that of Theorems 4.5 and 4.6, a brief outline is presented. Let

$$F(s,z) = \Gamma\left(\frac{s}{2} + \frac{z}{4}\right)\Gamma\left(\frac{1}{2} - \frac{s}{2} + \frac{z}{4}\right), \tag{5.93}$$

apply Theorem 5.3 to find by means of contour integration that $f(x,z) = \mathfrak{F}(x,z)$, as in (4.17). Then as before, Corollary 5.4 gives the transformation in (4.18). Also, starting from the integral on the extreme right of (4.18), converting it into a complex integral using (5.35), and then using the residue theorem and Mellin transforms, one finds this integral to be equal to one of the two sides of the transformation in (4.18). The transformation itself is then obtained by replacing α by β in this integral involving the Riemann Ξ function.

6. A generalization of a series of Koshlyakov

Koshlyakov [18] considered the function

$$\Omega(x) = 2\sum_{n=1}^{\infty} d(n)\left[K_0\left(4\pi e^{i\pi/4}\sqrt{nx}\right) + K_0\left(4\pi e^{-i\pi/4}\sqrt{nx}\right)\right] \tag{6.1}$$

and used it to give a short and clever proof of the Voronoï summation formula. In [18, Equation 5], he established the identity

$$\Omega(x) = -\gamma - \frac{1}{2}\log x - \frac{1}{4\pi x} + \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{d(n)}{x^2+n^2}. \tag{6.2}$$

The reader will find in [10] how to establish (6.2) using (4.14). Koshlyakov [20, Equation (6)], [22, Equations (21), (27)] showed that

$$\begin{aligned}\sqrt{\alpha}\int_0^{\infty} e^{-2\pi\alpha x}\left(\Omega(x) + \frac{1}{4\pi x}\right)dx &= \sqrt{\beta}\int_0^{\infty} e^{-2\pi\beta x}\left(\Omega(x) + \frac{1}{4\pi x}\right)dx \\ &= \frac{1}{2\pi^{5/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt.\end{aligned} \tag{6.3}$$

See [10] for the proof of the equivalence of the formula in Theorem 3.4 and (6.3) without appealing to the integral involving the Riemann Ξ-function in the identities. Koshlyakov [19, Equations (8), (24)], [21, Equation (15)] also found another transformation involving $\Omega(x)$, namely,

$$\begin{aligned}\sqrt{\alpha^3}\int_0^{\infty} xJ_0(2\pi\alpha x)\left(\Omega(x) + \frac{1}{4\pi x}\right)dx &= \sqrt{\beta^3}\int_0^{\infty} xJ_0(2\pi\beta x)\left(\Omega(x) + \frac{1}{4\pi x}\right)dx \\ = \frac{1}{64\pi^5}\int_0^{\infty}\left|\Gamma^2\left(\frac{-1+it}{4}\right)\right|^2\Xi^2\left(\frac{t}{2}\right)&\cosh\left(\frac{1}{2}\pi t\right)\cos\left(\tfrac{1}{2}t\log\alpha\right)dt.\end{aligned} \tag{6.4}$$

A new generalization of $\Omega(x)$, defined by

$$\Omega(x,z) = 2\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}\left(e^{\pi iz/4}K_z(4\pi e^{\pi i/4}\sqrt{nx}) + e^{-\pi iz/4}K_z(4\pi e^{-\pi i/4}\sqrt{nx})\right), \tag{6.5}$$

is considered next. It is clear that $\Omega(x,0) = \Omega(x)$. The inverse Mellin transform

$$\Omega(x,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(1-s+\frac{z}{2})\zeta(1-s-\frac{z}{2})}{2\cos\left(\frac{1}{2}\pi\left(s+\frac{z}{2}\right)\right)}x^{-s}ds \tag{6.6}$$

is validd for $c = \text{Re } s > 1 \pm \text{ Re } \frac{z}{2}$. The special case $z=0$ was given by Koshlyakov [21, Equation (11)] and the details for deriving the general case are similar to this special case.

The function $\Omega(x,z)$ plays an important role in deriving a simpler proof of the generalization of the Voronoï summation formula. See [4] for details.

Proposition 6.1. *For $z \notin \mathbb{Z}$, the function $\Omega(x,z)$ is given by*
(6.7)

$$\Omega(x,z) = -\frac{\Gamma(z)\zeta(z)}{(2\pi\sqrt{x})^z} + \frac{x^{z/2-1}}{2\pi}\zeta(z) - \frac{x^{z/2}}{2}\zeta(z+1) + \frac{x^{z/2+1}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{-z}(n)}{n^2+x^2}.$$

Actually, the above result is also true for $z = 0$ (in the limiting sense) in view of (6.2) and the fact that $\Omega(x,z)$ is continuous at $z = 0$.

The result in the proposition can be directly derived from the next theorem of H. Cohen [5]. Simply take $k = 1$ and replace x by ix and then by $-ix$ and add the results.

Theorem 6.2 (H. Cohen)**.** *Let $z \notin \mathbb{Z}$ be such that* $\mathrm{Re}\, z \geq 0$*. For any integer* $k \geq \left\lfloor \frac{1}{2}(\mathrm{Re}\, z + 1)\right\rfloor$,

$$8\pi x^{z/2}\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_z(4\pi\sqrt{nx}) = A(z,x)\zeta(z) + B(z,x)\zeta(z+1)$$

$$+\frac{2}{\sin(\pi z/2)}\left(\sum_{1\leq j\leq k}\zeta(2j)\zeta(2j-z)x^{2j-1} + x^{2k+1}\sum_{n=1}^{\infty}\sigma_{-z}(n)\frac{n^{z-2k}-x^{z-2k}}{n^2-x^2}\right),$$

where

$$A(z,x) = \frac{x^{z-1}}{\sin(\pi z/2)} - \frac{\Gamma(z)}{(2\pi)^{z-1}} \text{ and } B(z,x) = \frac{2}{x}\frac{\Gamma(z+1)}{(2\pi)^{z+1}} - \frac{\pi x^z}{\cos(\pi z/2)}.$$

A generalization of (6.3) is stated next.

Theorem 6.3. *Assume $-1 < \mathrm{Re}\, z < 1$. Then for $\alpha, \beta > 0, \alpha\beta = 1$,*

$$\alpha^{(z+1)/2}\int_0^{\infty} e^{-2\pi\alpha x}x^{z/2}\left(\Omega(x,z) - \frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx$$
$$= \beta^{(z+1)/2}\int_0^{\infty} e^{-2\pi\beta x}x^{z/2}\left(\Omega(x,z) - \frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx$$
$$= \frac{8}{\pi^{(z+5)/2}}\int_0^{\infty}\Gamma\left(\frac{z+3+it}{4}\right)\Gamma\left(\frac{z+3-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)$$
$$\times\frac{\cos(\frac{1}{2}t\log\alpha)\,dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}.$$

Proof. The theorem is proven first for $0 <$ Re $z < 1$ and later extend it to $-1 <$ Re $z < 1$ by analytic continuation. Using (5.35) with $\phi(z,s) = \frac{1}{2s+z+1}\Gamma\left(\frac{z-1}{4}+\frac{s}{2}\right)$ and the reflection and the duplication formulas for the gamma function, the integral on the extreme right above (say $M(z,\alpha)$) can be written as
(6.8)

$$M(z,\alpha) := \frac{\sqrt{\pi}}{2^{1+\frac{z}{2}}i\sqrt{\alpha}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(1-s+\frac{z}{2}\right)\frac{\zeta\left(1-s+\frac{z}{2}\right)\zeta\left(1-s-\frac{z}{2}\right)}{2\cos\left(\frac{1}{2}\pi\left(s+\frac{z}{2}\right)\right)}\left(\frac{1}{2\pi\alpha}\right)^{-s}ds.$$

To use (5.5), one needs to evaluate the inverse Mellin transforms of the two functions, namely $\zeta\left(1-s+\frac{z}{2}\right)\zeta\left(1-s-\frac{z}{2}\right)/\left(2\cos\left(\frac{1}{2}\pi\left(s+\frac{z}{2}\right)\right)\right)$ and $\Gamma\left(1-s+\frac{z}{2}\right)$ in a common region which includes the vertical line Re $s=\frac{1}{2}$.

For $c = \mathrm{Re}\, s < 1 \pm \mathrm{Re}\left(\frac{z}{2}\right)$, using (6.6) and invoking the residue theorem results in

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta\left(1-s+\frac{z}{2}\right)\zeta\left(1-s-\frac{z}{2}\right)}{2\cos\left(\frac{1}{2}\pi\left(s+\frac{z}{2}\right)\right)}x^{-s}\,ds=\Omega(x,z)-\frac{\zeta(z)}{2\pi}x^{\frac{z}{2}-1}. \tag{6.9}$$

Also for $c=$ Re $s<1+$ Re $\left(\frac{z}{2}\right)$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma\left(1-s+\frac{z}{2}\right)x^{-s}\,ds=e^{-\frac{1}{x}}x^{-1-\frac{z}{2}}. \tag{6.10}$$

Since $0<$ Re $z<1$, shifting the line of integration to $c=\mathrm{Re}\, s<1-\mathrm{Re}\left(\frac{z}{2}\right)$ does not introduce a pole. Therefore the above formula is valid for $c=\mathrm{Re}\, s<1\pm\mathrm{Re}\left(\frac{z}{2}\right)$. Also, the choice $c=\frac{1}{2}$ is valid since $0<\mathrm{Re}\, z<1$. Thus employing (5.5), gives

$$\begin{aligned}&\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(1-s+\frac{z}{2}\right)\frac{\zeta\left(1-s+\frac{z}{2}\right)\zeta\left(1-s-\frac{z}{2}\right)}{2\cos\left(\frac{1}{2}\pi\left(s+\frac{z}{2}\right)\right)}\left(\frac{1}{2\pi\alpha}\right)^{-s}ds\\ &=2\pi i\int_0^\infty e^{-2\pi\alpha x}(2\pi\alpha x)^{1+\frac{z}{2}}\left(\Omega(x,z)-\frac{\zeta(z)}{2\pi}x^{\frac{z}{2}-1}\right)\frac{dx}{x}.\end{aligned} \tag{6.11}$$

Therefore (6.8) and (6.11) establish the equality between the extreme left and right sides of Theorem 6.3. Now using (5.24) and known bounds on the Riemann zeta function, it is easy to see that $M(z,\alpha)$ is analytic in $-1<$ Re $z<1$. Similarly, using Proposition 6.1, it is easy to see that the extreme left side of Theorem 6.3 is analytic in $-1<$ Re $z<1$. Thus by analytic continuation, the equality holds for $-1<$ Re $z<1$. Now, as usual, replace α by β in the established identity and use the relation $\alpha\beta=1$ to obtain the second equality in Theorem 6.3. □

A generalization of (6.4) is stated next.

Theorem 6.4. *Assume* $-1<\mathrm{Re}\, z<1$. *Then for* $\alpha,\beta>0,\alpha\beta=1$,

$$\begin{aligned}&\sqrt{\alpha^3}\int_0^\infty xJ_{z/2}(2\pi\alpha x)\left(\Omega(x,z)-\frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx\\ &=\sqrt{\beta^3}\int_0^\infty xJ_{z/2}(2\pi\beta x)\left(\Omega(x,z)-\frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx\\ &=\frac{8}{\pi^3}\int_0^\infty\frac{\Gamma\left(\frac{z+3+it}{4}\right)\Gamma\left(\frac{z+3-it}{4}\right)}{\Gamma\left(\frac{z+1+it}{4}\right)\Gamma\left(\frac{z+1-it}{4}\right)}\Xi\left(\frac{t-iz}{2}\right)\Xi\left(\frac{t+iz}{2}\right)\\ &\qquad\times\frac{\cos(\frac{1}{2}t\log\alpha)\,dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}.\end{aligned}$$

Proof. The proof is based on the Mellin transform (5.12), the inverse Mellin transform (6.9) and (5.4). Details are omitted. □

Note 6.5. The integral on the right-hand side of Theorem 6.3 appears in S. Ramanujan [31, Equation (20)] and in [8, Theorem 1.5], where alternate representations for this integral have been given. Comparing the representation derived here with these, lead to the following identity:

$$\begin{aligned}(6.12)\quad &\alpha^{(z+1)/2}\int_0^\infty e^{-2\pi\alpha x}x^{z/2}\left(\Omega(x,z)-\frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx\\ &=\frac{1}{(2\pi)^{z+1}}\int_0^\infty x^z\left(\frac{1}{e^{x\sqrt{\alpha}}-1}-\frac{1}{x\sqrt{\alpha}}\right)\left(\frac{1}{e^{x/\sqrt{\alpha}}-1}-\frac{1}{x/\sqrt{\alpha}}\right)dx\\ &=\alpha^{(z+1)/2}\frac{\Gamma(z+1)}{(2\pi)^{z+1}}\left[\sum_{n=1}^\infty\left(\zeta(z+1,n\alpha)-\frac{(n\alpha)^{-z}}{z}-\frac{(n\alpha)^{-z-1}}{2}\right)-\frac{\zeta(z+1)}{2\alpha^{z+1}}-\frac{\zeta(z)}{\alpha z}\right],\end{aligned}$$

where $\zeta(z,x)$ is the Hurwitz zeta function.

7. Related work of Guinand and of Nasim

The work presented here is related to results of Guinand and Nasim. These are presented next. Guinand [13, Theorem 6] (see also [14, Equation (1)] obtained the following summation formula involving $\sigma_s(n)$:

$$\begin{aligned}&\sum_{n=1}^\infty\sigma_{-s}(n)n^{\frac{s}{2}}f(n)-\zeta(1+s)\int_0^\infty x^{\frac{s}{2}}f(x)\,dx-\zeta(1-s)\int_0^\infty x^{-\frac{s}{2}}f(x)\,dx\\ (7.1)\quad &=\sum_{n=1}^\infty\sigma_{-s}(n)n^{\frac{s}{2}}g(n)-\zeta(1+s)\int_0^\infty x^{\frac{s}{2}}g(x)\,dx-\zeta(1-s)\int_0^\infty x^{-\frac{s}{2}}g(x)\,dx.\end{aligned}$$

Here $f(x)$ satisfies appropriate conditions (see [13] for details) and $g(x)$ is the transform of $f(x)$ with respect to the Fourier kernel

$$(7.2)\quad -2\pi\sin\left(\tfrac{1}{2}\pi s\right)J_s(4\pi\sqrt{x})-\cos\left(\tfrac{1}{2}\pi s\right)\left(2\pi Y_s(4\pi\sqrt{x})-4K_s(4\pi\sqrt{x})\right).$$

Note that up to a constant factor, the above kernel is the same as the one used in (5.6). C. Nasim [25, 26, 27] also derived transformation formulas similar to (7.1).

As an application of (7.1), note that for z fixed and $-1<\text{Re } z<1$, one obtains (4.3) by taking $f(x)=K_{\frac{z}{2}}(2\pi\alpha x)$, and then using (5.6) with $\nu=z/2$ and (5.17). This is the simplest example of (7.1), since here $f(x)=g(x)$. The disadvantage of (7.1) is the difficulty in obtaining other explicit examples. Note that Guinand [14] does not give any particular example of (7.1)[3]. The production of a function g requires the explicit evaluation of the integral

$$\int_0^\infty f(y)\left(-2\pi\sin\left(\tfrac{1}{2}\pi s\right)J_s(4\pi\sqrt{xy})-\cos\left(\tfrac{1}{2}\pi s\right)\left(2\pi Y_s(4\pi\sqrt{xy})-4K_s(4\pi\sqrt{xy})\right)\right)dy.$$

[3]Guinand's proof of (4.3) is different from the one given above.

Instead, considering transformations between two integrals, as done here, one can construct a variety of explicit examples. These are shown in Theorems 4.5, 4.6 and 4.9. Also in order to explicitly find a transformation using (7.1), one has to start with a proper f. A priori, it is hard to conceive of an f which would lead to an elegant transformation. The advantage in our case is that the results are obtained by considering known transformation formulas in the literature. Thus, one does not have to begin with an unnatural choice of f.

8. Future developments

The results presented here deal with modular-type transformations which result from squaring the functional equation of $\zeta(s)$, or equivalently, those, whose associated integrals involving the Riemann Ξ-functions have $\frac{\Xi^2(t/2)}{(1+t^2)^2}$ in their integrand. These can be extended by taking higher powers of the functional equation, or equivalently, by taking $\frac{\Xi^m(t/2)}{(1+t^2)^m}$ in the integrand of the associated integrals. The consequences of this extension are discussed in a particular example.

In view of Hardy's result (4.5) and Koshlyakov's result (2.12), it is also possible to evaluate the integral

$$\int_0^{\infty} \left(\frac{\Xi(t/2)}{(1+t^2)} \right)^m \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{\cosh\left(\frac{1}{2}\pi t\right)}\, dt,$$

and obtain the corresponding modular-type transformation. Note that when $m = 1$, the transformation involves the series

$$-\psi(x+1) - \gamma = \sum_{n=1}^{\infty} \left(\frac{1}{x+n} - \frac{1}{n} \right)$$

and when $m = 2$, the series

$$\sum_{n=1}^{\infty} d(n) \left(\frac{1}{x+n} - \frac{1}{n} \right).$$

Thus for $m > 2$, the transformation would involve the series

$$\sum_{n=1}^{\infty} d_m(n) \left(\frac{1}{x+n} - \frac{1}{n} \right),$$

where $d_m(n)$ denotes the number of ways of expressing n as a product of m factors in which an expression with the same factors but in a different order is counted as different. The Dirichlet series for $d_m(n)$ is given by

$$\zeta^m(s) = \sum_{n=1}^{\infty} \frac{d_m(n)}{n^s}. \tag{8.1}$$

The series $\sum_{n=1}^{\infty} d_m(n)\left(\frac{1}{x+n}-\frac{1}{n}\right)$ as well as the residues associated with it come from evaluating the integral

$$\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\frac{\pi}{\sin \pi s}\zeta^m(1-s)x^{-s}\,ds. \tag{8.2}$$

Let f be the function against which the above expression is integrated in order to obtain the transformation. In the case $m=1$, $f(x)=e^{-\pi\alpha^2x^2}$ whose Mellin transform is $\frac{1}{2}\pi^{-s/2}\alpha^{-s}\Gamma\left(\frac{s}{2}\right)$, and when $m=2$, $f(x)=K_0(2\pi\alpha x)$ whose Mellin transform is $\frac{1}{4}\pi^{-s}\alpha^{-s}\Gamma^2\left(\frac{s}{2}\right)$. Thus, for $m>2$, the Mellin transform of f should be a multiple of $\Gamma^m\left(\frac{s}{2}\right)$. These general transformations will be discussed in a future publication.

Acknowledgments. The second author acknowledges the partial support of NSF-DMS 1112656. The first author is a post-doctoral fellow, funded in part by the same grant.

References

[1] T. Apostol. *Introduction to Analytic Number Theory.* Springer-Verlag, New York, 1976.

[2] B. Berndt and A. Dixit. A transformation formula involving the Gamma and Riemann zeta functions in Ramanujan's Lost Notebook. In C. R. Rao K. Alladi, J. Klauder, editor, *The legacy of Alladi Ramakrishnan in the mathematical sciences*, pages 199–210. Springer-Verlag, New York, 2010.

[3] B. Berndt, Y. Lee, and J. Sohn. Koshliakov's formula and Guinand's formula in Ramanujan's Lost Notebook. In K. Alladi, editor, *Surveys in Number Theory*, volume 17 of *Series: Development in Mathematics*, pages 21–42. Springer-Verlag, New York, 2008.

[4] B. C. Berndt, A. Dixit, A. Roy, and A. Zaharescu. A series identity, possibly connected with a divisor problem, in Ramanujan's Lost Notebook. *In preparation*, 2013.

[5] H. Cohen. Some formulas of Ramanujan involving Bessel functions. In Algèbre et Théorie des Nombres, editor, *Publications Mathèmatiques de Besançon*, pages 59–68, 2010.

[6] A. Dixit. Series transformations and integrals involving the Riemann ξ-function. *Jour. Math. Appl.*, 368:358–373, 2010.

[7] A. Dixit. Analogues of a transformation formula of Ramanujan. *Intern. J. Number Theory*, 7:1151–1172, 2011.

[8] A. Dixit. Transformation formulas associated with integrals involving the Riemann Ξ-function. *Monatsh. Math.*, 164:133–156, 2011.

[9] A. Dixit. Analogues of the general theta transformation formula. *Proc. Roy. Soc. Edinburgh, Sect. A*, 143:371–399, 2013.

[10] A. Dixit. Ramanujan's ingenious method for generating modular-type transformation formulas. In *Proceedings of the Legacy of Ramanujan conference.* University of Delhi, Ramanujan Mathematical Society, 2013.

[11] W. L. Ferrar. Some solutions of the equation $F(t)=F(t^{-1})$. *Jour. London Math. Soc.*, 11:99–103, 1936.

[12] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.

[13] A. P. Guinand. Summation formulae and self-reciprocal functions II. *Quart. J. Math.*, 10:104–118, 1939.

[14] A. P. Guinand. Some rapidly convergent series for the Riemann ξ-function. *Quart. J. Math.*, 6:156–160, 1955.
[15] G. H. Hardy. Note by Mr. G. H. Hardy on the preceding paper. *Quart. J. Math.*, 46:260–261, 1915.
[16] G. H. Hardy and E. C. Titchmarsh. Self-reciprocal functions. *Quart. J. Math.*, 1:196–231, 1930.
[17] C. G. J. Jacobi. Fundamenta nova theoriae functionum ellipticarum. In *Gesammelte Werke*, volume I, pages 49–239. Chelsea Publishing Company, 1829.
[18] N. S. Koshlyakov. On Voronoi's sum formula. *Mess. Math.*, 58:30–32, 1929.
[19] N. S. Koshlyakov. Some integral representations of the square of Riemann's function $\Xi(t)$. *Dokl. Akad. Nauk SSSR*, 2:401–405, 1934.
[20] N. S. Koshlyakov. Note on some infinite integrals. *Comp. Rend. (Doklady) Acad. Sci. URSS*, 2:247–250, 1936.
[21] N. S. Koshlyakov. On an extension of some formulae of Ramanujan. *Proc. London Math, Soc.*, 41:26–32, 1936.
[22] N. S. Koshlyakov. On a transformation of definite integrals and its application to the theory of Riemann's function $\xi(s)$. *Comp. Rend. (Doklady) Acad. Sci. URSS*, 15:3–8, 1937.
[23] N. S. Koshlyakov. Notes on certain integrals involving Bessel functions. *Bull. Acad. Sci. URSS Ser. Math.*, 2:417–420, 1938.
[24] H. P. McKean and V. Moll. *Elliptic Curves: Function Theory, Geometry, Arithmetic*. Cambridge University Press, New York, 1997.
[25] C. Nasim. A summation formula involving $\sigma_k(n)$, $k > 1$. *Canad. J. Math.*, 21:951–964, 1969.
[26] C. Nasim. On the summation formula of Voronoï. *Trans. Amer. Math. Soc.*, 163:35–45, 1971.
[27] C. Nasim. A summation formula involving $\sigma(n)$. *Trans. Amer. Math. Soc.*, 192:307–317, 1974.
[28] F. Oberhettinger. *Tables of Mellin transforms*. Springer-Verlag, New York, 1974.
[29] R. B. Paris and D. Kaminski. *Asymptotics and Mellin-Barnes Integrals*, volume 85 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, 2001.
[30] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and Series*, volume 2: Special functions. Gordon and Breach Science Publishers, 1986.
[31] S. Ramanujan. New expressions for Riemann's functions $\xi(s)$ and $\Xi(t)$. *Quart. J. Math.*, 46:253–260, 1915.
[32] S. Ramanujan. *The Lost Notebooks and Other Unpublished Papers*. Narosa, New Delhi, 1988.
[33] E. C. Titchmarsh. *The theory of the Riemann zeta function*. Oxford University Press, 2nd edition, 1986.
[34] G. Voronoï. Sur une fonction transcendante et ses applications à la sommation de quelques séries. *Annales de l'Ecole Normale, Ser. 3*, 21:207–267, 1904.

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: adixit@tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@tulane.edu