SERIES TRANSFORMATIONS AND INTEGRALS INVOLVING THE RIEMANN Ξ-FUNCTION

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Abstract. The transformation formulas of Ramanujan, Hardy, Koshliakov and Ferrar are unified, in the sense that all these formulas come from the same source, namely, a general formula involving an integral of Riemann's Ξ-function. We give proofs of all of these transformation formulas using the theory of Mellin transforms and the residue theorem. Our study includes new extensions of the formulas of Koshliakov and Ferrar through their connection with integrals involving the Riemann Ξ-function.

Key Words: Riemann zeta function, Riemann Ξ-function, psi function, modified Bessel function, Residue theorem, Mellin transform, Ramanujan, Hardy.

1. Introduction

In the year 1929, N.S. Koshliakov [13] discovered a result now remembered as Koshliakov's formula. To state his theorem, let $K_\nu(z)$ denote the modified Bessel function of order ν, and let $d(n)$ denote the number of positive divisors of the positive integer n. Then, if γ denotes Euler's constant and $a > 0$,

$$\gamma - \log\left(\frac{4\pi}{a}\right) + 4\sum_{n=1}^{\infty} d(n)K_0(2\pi an) = \frac{1}{a}\left(\gamma - \log(4\pi a) + 4\sum_{n=1}^{\infty} d(n)K_0\left(\frac{2\pi n}{a}\right)\right). \tag{1.1}$$

Later in 1936, W.L. Ferrar [6] showed that Koshliakov's formula is equivalent to the functional equation for $\zeta^2(s)$, where $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.2}$$

Ferrar rephrased Koshliakov's formula in the form $F(\alpha) = F(\beta)$, where $\alpha\beta = 1$, given below.

Theorem 1.1. *If $K_\nu(z)$, $d(n)$ and γ are defined as before and if α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\sqrt{\alpha}\left(\frac{\gamma - \log(4\pi\alpha)}{\alpha} - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\alpha)\right) = \sqrt{\beta}\left(\frac{\gamma - \log(4\pi\beta)}{\beta} - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\beta)\right). \tag{1.3}$$

Actually Ramanujan had discovered Koshliakov's formula before Koshliakov, as can be seen from page 253 of Ramanujan's Lost Notebook [20]. See [3] for details.

2000 *Mathematics Subject Classification.* Primary 11M06.

In the same paper [6], Ferrar demonstrated some other solutions of the general equation $F(\alpha) = F(\beta)$, where $\alpha\beta = 1$. For example,

Theorem 1.2. *If α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\sqrt{\alpha}\left(-\gamma + \log 16\pi - 2\log\alpha + 2\sum_{n=1}^{\infty}\left(e^{\frac{\pi\alpha^2n^2}{2}}K_0\left(\frac{\pi\alpha^2n^2}{2}\right) - \frac{1}{n\alpha}\right)\right)$$
$$= \sqrt{\beta}\left(-\gamma + \log 16\pi - 2\log\beta + 2\sum_{n=1}^{\infty}\left(e^{\frac{\pi\beta^2n^2}{2}}K_0\left(\frac{\pi\beta^2n^2}{2}\right) - \frac{1}{n\beta}\right)\right). \tag{1.4}$$

Ferrar's method in [6] is general in the sense that it applies to any Dirichlet series having a functional equation. However in this paper, we would like to emphasize an alternative method for producing solutions of the equation $F(\alpha) = F(\beta)$ for $\alpha\beta = 1$ through a connection with an integral involving the Riemann Ξ-function, the prototype of which can be found in a manuscript of Ramanujan, in the handwriting of G.N. Watson, contained in the Lost Notebook [20, p. 220] written several years before the papers of Koshliakov and Ferrar. Ramanujan's beautiful claim is as follows.

Theorem 1.3. *Define*

$$\phi(x) := \psi(x) + \frac{1}{2x} - \log x, \tag{1.5}$$

where

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{m=0}^{\infty}\left(\frac{1}{m+x} - \frac{1}{m+1}\right), \tag{1.6}$$

the logarithmic derivative of the Gamma function. Let Riemann's ξ-function be defined by

$$\xi(s) := (s-1)\pi^{-\frac{1}{2}s}\Gamma(1+\tfrac{1}{2}s)\zeta(s), \tag{1.7}$$

and let

$$\Xi(t) := \xi(\tfrac{1}{2} + it) \tag{1.8}$$

be the Riemann Ξ-function. If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\phi(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\phi(n\beta)\right\}$$
$$= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt, \tag{1.9}$$

where γ here again denotes Euler's constant.

A.P. Guinand [8, 9] rediscovered the first equality in (1.9) in a slightly different form. Recently, B.C. Berndt and A. Dixit [2] proved both parts of (1.9). Later, A. Dixit [5] obtained (1.9) as a limiting case of a more general formula given below.

Theorem 1.4. *Let $0 < \text{Re } z < 2$. Define $\varphi(z, x)$ by*

$$\varphi(z, x) := \zeta(z, x) - \frac{1}{2}x^{-z} + \frac{x^{1-z}}{1-z}, \tag{1.10}$$

where $\zeta(z,x)$ *denotes the Hurwitz zeta function. Then if* α *and* β *are any positive numbers such that* $\alpha\beta=1$,

$$\alpha^{\frac{z}{2}}\left(\sum_{n=1}^{\infty}\varphi(z,n\alpha)-\frac{\zeta(z)}{2\alpha^{z}}-\frac{\zeta(z-1)}{\alpha(z-1)}\right)=\beta^{\frac{z}{2}}\left(\sum_{n=1}^{\infty}\varphi(z,n\beta)-\frac{\zeta(z)}{2\beta^{z}}-\frac{\zeta(z-1)}{\beta(z-1)}\right)$$
$$=\frac{8(4\pi)^{\frac{z-4}{2}}}{\Gamma(z)}\int_0^{\infty}\Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt, \tag{1.11}$$

where $\Xi(t)$ *is defined in* (1.8)*.*

Ramanujan's transformation formula (1.9) not only gives an example of F satisfying $F(\alpha)=F(\beta)$, where $\alpha\beta=1$, but also reveals a nice connection with an integral involving the Riemann Ξ-function. This suggests that one might try to find such integral representations for Theorems 1.1 and 1.2. These representations which extend the formulas of Koshliakov and Ferrar are derived in Sections 2 and 4 and are as follows.

Theorem 1.5 (Extended version of Koshliakov's formula)**.** *If* α *and* β *are positive numbers such that* $\alpha\beta=1$, *then*

$$\sqrt{\alpha}\left(\frac{\gamma-\log(4\pi\alpha)}{\alpha}-4\sum_{n=1}^{\infty}d(n)K_0(2\pi n\alpha)\right)=\sqrt{\beta}\left(\frac{\gamma-\log(4\pi\beta)}{\beta}-4\sum_{n=1}^{\infty}d(n)K_0(2\pi n\beta)\right)$$
$$=-\frac{32}{\pi}\int_0^{\infty}\frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2\cos\left(\frac{1}{2}t\log\alpha\right)\,dt}{(1+t^2)^2}. \tag{1.12}$$

Theorem 1.6 (Extended version of Ferrar's formula)**.** *If* α *and* β *are positive numbers such that* $\alpha\beta=1$, *then*

$$\sqrt{\alpha}\left(\frac{-\gamma+\log 16\pi+2\log\alpha}{\alpha}-2\sum_{n=1}^{\infty}\left(e^{\frac{\pi\alpha^2n^2}{2}}K_0\left(\frac{\pi\alpha^2n^2}{2}\right)-\frac{1}{n\alpha}\right)\right)$$
$$=\sqrt{\beta}\left(\frac{-\gamma+\log 16\pi+2\log\beta}{\beta}-2\sum_{n=1}^{\infty}\left(e^{\frac{\pi\beta^2n^2}{2}}K_0\left(\frac{\pi\beta^2n^2}{2}\right)-\frac{1}{n\beta}\right)\right)$$
$$=4\pi^{-\frac{3}{2}}\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt. \tag{1.13}$$

Two further examples of a transformation formula and an integral involving the Riemann Ξ-function associated with it, namely equations (1.15) and (1.18), can be easily derived from Ramanujan's formula (Equation (1.14) below) and Hardy's formula (Equation (1.17) below) respectively. In [17], Ramanujan derives the identity for real n,

$$e^{-n}-4\pi e^{-3n}\int_0^{\infty}\frac{xe^{-\pi x^2e^{-4n}}}{e^{2\pi x}-1}\,dx=\frac{1}{4\pi\sqrt{\pi}}\int_0^{\infty}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\cos nt\,dt. \tag{1.14}$$

Letting $n=\frac{1}{2}\log\alpha$ in (1.14) and noting that the integral on the right-hand side is invariant under the map $\alpha\to\beta$, where $\alpha\beta=1$, we deduce the following result.

Theorem 1.7. *If α and β are two positive numbers such that $\alpha\beta = 1$, then*

$$\alpha^{-\frac{1}{2}} - 4\pi\alpha^{-\frac{3}{2}} \int_0^\infty \frac{xe^{-\frac{\pi x^2}{\alpha^2}}}{e^{2\pi x}-1}\,dx = \beta^{-\frac{1}{2}} - 4\pi\beta^{-\frac{3}{2}} \int_0^\infty \frac{xe^{-\frac{\pi x^2}{\beta^2}}}{e^{2\pi x}-1}\,dx$$
$$= \frac{1}{4\pi\sqrt{\pi}} \int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt. \tag{1.15}$$

The first equality in the above formula can be easily seen to be equivalent to the following well-known identity of Ramanujan.

Theorem 1.8. *If α and β be any two positive numbers such that $\alpha\beta = \pi^2$, then*

$$\alpha^{-\frac{1}{4}}\left(1 + 4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right) = \beta^{-\frac{1}{4}}\left(1 + 4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right). \tag{1.16}$$

Ramanujan discussed (1.16) in [17], just after proving (1.14). Another proof of this identity can be seen in a paper of Ramanujan [18]. It also appears in Ramanujan's first letter to Hardy [19, p. xxvi] (Chelsea reprint). Further, this result was also established by C. T. Preece [16]. See [1, p. 291] for more details.

Another example of such a function F can be easily derived from an identity found in a 1915 paper of G.H. Hardy [11] (see (1.17) below) in the Quarterly Journal of Mathematics, immediately following Ramanujan's paper [17]. Interestingly, this short note is not reproduced in any of the seven volumes of the Collected Papers of G.H. Hardy (see [10, pp. 691-692] for example). In this note, Hardy says that the integral on the right-hand side in Ramanujan's formula (1.14) can be used to prove Hardy's result that there are infinitely many zeros of $\zeta(s)$ on the critical line Re $s = \frac{1}{2}$, and then he concludes the note by giving (1.17) below, which he says is not unlike (1.14). However, Hardy does not give a proof of his formula, and no proof has been supplied later by anyone else either. It turns out that there is a small error in the original formula given by Hardy. The sign of one of the expressions in it, namely of $\dfrac{\gamma}{2}$, should be $+$ and not $-$. We will sketch a proof of this formula in Section 5.

Theorem 1.9 (Correct version of Hardy's claim)**.** *For n real, we have*

$$\int_0^\infty \frac{\Xi(\frac{1}{2}t)}{1+t^2}\frac{\cos nt}{\cosh\frac{1}{2}\pi t}\,dt = \frac{1}{4}e^{-n}\left(2n + \frac{1}{2}\gamma + \frac{1}{2}\log\pi + \log 2\right) + \tfrac{1}{2}e^{n}\int_0^\infty \psi(x+1)e^{-\pi x^2 e^{4n}}\,dx, \tag{1.17}$$

where $\psi(x)$ is defined in (1.6)*.*

Now letting $n = \frac{1}{2}\log\alpha$ in (1.17) and noting that the integral on the right-hand side is invariant under the map $\alpha \to \beta$, where $\alpha\beta = 1$, we have another example of a function F satisfying $F(\alpha) = F(\beta)$, where $\alpha\beta = 1$.

Theorem 1.10. *If α and β are two positive numbers such that $\alpha\beta = 1$, then*

$$\begin{aligned}&\frac{\sqrt{\alpha}}{2}\int_0^\infty \psi(x+1)e^{-\pi\alpha^2x^2}\,dx + \frac{1}{4\sqrt{\alpha}}(\log\alpha + \tfrac{1}{2}\gamma + \tfrac{1}{2}\log\pi + \log 2)\\ &= \frac{\sqrt{\beta}}{2}\int_0^\infty \psi(x+1)e^{-\pi\beta^2x^2}\,dx + \frac{1}{4\sqrt{\beta}}(\log\beta + \tfrac{1}{2}\gamma + \tfrac{1}{2}\log\pi + \log 2)\\ &= \int_0^\infty \frac{\Xi(\frac{1}{2}t)}{1+t^2}\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{\cosh\frac{1}{2}\pi t}\,dt. \end{aligned}\tag{1.18}$$

This paper is organized as follows. In Section 2, we prove Theorem 1.5. Then in Section 3, we give another proof of Theorem 1.3 different from the ones given in [2] and [5]. In Sections 4 and 5, we give only brief sketches of the proofs of Theorems 1.6 and 1.9 respectively, since the same method using the theory of Mellin transforms and the residue theorem is employed to derive all four of these formulas. The common source for proving them is a simple formula found in [22, p. 35], which we derive here to make this paper self-contained. Let

$$f(t) = |\phi(it)|^2 = \phi(it)\phi(-it), \tag{1.19}$$

where ϕ is analytic. Also let $y = e^n$ for n real. Then,

$$\begin{aligned}\int_0^\infty f(t)\Xi(t)\cos nt\,dt &= \frac{1}{2}\int_{-\infty}^\infty \phi(it)\phi(-it)\Xi(t)y^{it}\,dt\\ &= \frac{1}{2}\int_{-\infty}^\infty \phi(it)\phi(-it)\xi\left(\frac{1}{2}+it\right)y^{it}\,dt\\ &= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi(s-\tfrac{1}{2})\phi(\tfrac{1}{2}-s)\xi(s)y^s\,ds. \end{aligned}\tag{1.20}$$

Actually we use (1.20) in a slightly different form. Replacing t by $t/2$ on the left-hand side of (1.20) and then replacing n by $2n$ in (1.20) yields

$$\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)\cos nt\,dt = \frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi(s-\tfrac{1}{2})\phi(\tfrac{1}{2}-s)\xi(s)y^s\,ds, \tag{1.21}$$

where $y = e^{2n}$. It is (1.21) that we will use in subsequent sections.

2. Koshliakov's formula

In this section, we prove Theorem 1.5. Even though the details in the latter part of the proof are similar to Ferrar's proof of Koshliakov's formula in [6], we give a complete proof so as to make this work self-contained. Let

$$f(t) := \frac{4\Xi(t)}{(\frac{1}{4}+t^2)^2}. \tag{2.1}$$

Then from (1.19) and the fact that $\xi(s) = \xi(1-s)$, where $\xi(s)$ is defined in (1.7), we find that

$$\phi(s) = \frac{2\sqrt{\xi(\frac{1}{2}-s)}}{(\frac{1}{2}+s)(\frac{1}{2}-s)}. \tag{2.2}$$

Thus from (1.21), (2.1) and (2.2), we see that

$$\begin{aligned}\int_0^\infty \frac{64\left(\Xi\left(\frac{t}{2}\right)\right)^2 \cos nt}{(1+t^2)^2}\, dt &= \frac{1}{i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{4\sqrt{\xi(s)}\sqrt{\xi(1-s)}}{s^2(s-1)^2} \xi(s) y^s\, ds \\ &= \frac{1}{i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{4\xi^2(s)}{s^2(s-1)^2} y^s\, ds \\ &= \frac{1}{i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s) y^s\, ds.\end{aligned} \tag{2.3}$$

Now to examine the integral in the last expression in (2.3), we wish to move the line of integration from $\text{Re } s = \frac{1}{2}$ to $\text{Re } s = 1+\delta$, for some $\delta > 0$, so that we can use the series representation for $\zeta^2(s)$ [22, p. 4], namely,

$$\zeta^2(s) = \sum_{m=1}^\infty \frac{d(m)}{m^s}. \tag{2.4}$$

But while doing that, we need to take care of the pole of order 2 (due to $\zeta(s)$) of the integrand at $s = 1$ in the last expression in (2.3).

Let $T > 0$ denote a real number. Then by the residue theorem, we know that

$$\begin{aligned}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s) y^s\, ds &= \left[\int_{\frac{1}{2}-iT}^{1+\delta-iT} + \int_{1+\delta-iT}^{1+\delta+iT} + \int_{1+\delta+iT}^{\frac{1}{2}+iT}\right] \pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s) y^s\, ds \\ &\quad - 2\pi i \lim_{s\to1} \frac{d}{ds}\left((s-1)^2\pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s) y^s\right).\end{aligned} \tag{2.5}$$

First, we evaluate $\lim_{s\to1} \frac{d}{ds}\left((s-1)^2\pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s) y^s\right)$. Using the product rule for differentiation and then simplifying, we have

$$\begin{aligned}&\frac{d}{ds}\left((s-1)^2\left(\frac{y}{\pi}\right)^s\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\right) \\ &= 2(s-1)\left(\frac{y}{\pi}\right)^s\Gamma^2\left(\frac{s}{2}\right)\zeta(s)\left(\zeta(s)+(s-1)\zeta'(s)\right) \\ &\quad + (s-1)^2\left(\frac{y}{\pi}\right)^s\log\left(\frac{y}{\pi}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s) + (s-1)^2\left(\frac{y}{\pi}\right)^s\Gamma\left(\frac{s}{2}\right)\Gamma'\left(\frac{s}{2}\right)\zeta^2(s) \\ &= f_1(s)+f_2(s)+f_3(s), \text{ say.}\end{aligned} \tag{2.6}$$

Since from [22, p. 20], we have

$$\zeta(s)=\frac{1}{s-1}+\gamma-\gamma_1(s-1)+\cdots, \tag{2.7}$$

we see that,

$$\lim_{s\to1}\Big(\zeta(s)+(s-1)\zeta'(s)\Big)=\gamma. \tag{2.8}$$

Hence from (2.6), we deduce that

$$\lim_{s\to1}f_1(s)=2\gamma y, \tag{2.9}$$

$$\lim_{s\to1}f_2(s)=y\log\left(\frac{y}{\pi}\right), \tag{2.10}$$

and

$$\begin{aligned}\lim_{s\to1}f_3(s)&=\lim_{s\to1}(s-1)^2\left(\frac{y}{\pi}\right)^s\Gamma^2\left(\frac{s}{2}\right)\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)\zeta^2(s)\\&=y(-\gamma-2\log2),\end{aligned} \tag{2.11}$$

since [7, p. 895, formula 8.366, no. 2]

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)=-\gamma-2\log2, \tag{2.12}$$

and $\Gamma\left(\frac12\right)=\sqrt{\pi}$. Hence from (2.6), (2.9), (2.10) and (2.11), we conclude that

$$\lim_{s\to1}\frac{d}{ds}\left((s-1)^2\left(\frac{y}{\pi}\right)^s\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\right)=y\left(\gamma-\log\left(\frac{4\pi}{y}\right)\right). \tag{2.13}$$

Using Stirling's formula on a vertical strip

$$|\Gamma(s)|=(2\pi)^{\frac12}|t|^{\sigma-\frac12}e^{-\frac12\pi|t|}\left(1+O\left(\frac{1}{|t|}\right)\right), \tag{2.14}$$

and the fact [22, p. 95] that for Re $s\geq\frac12$, we have

$$\zeta(s)=O(|t|), \tag{2.15}$$

one can easily observe that

$$\lim_{T\to\infty}\int_{\frac12\pm iT}^{1+\delta\pm iT}\pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)y^s\,ds=0. \tag{2.16}$$

Now it remains to evaluate $\int_{1+\delta-i\infty}^{1+\delta+i\infty}\pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)y^s\,ds$. Using (2.4), we observe that

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty}\pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)y^s\,ds=\sum_{m=1}^{\infty}d(m)\int_{1+\delta-i\infty}^{1+\delta+i\infty}\Gamma^2\left(\frac{s}{2}\right)\left(\frac{\pi m}{y}\right)^{-s}ds \tag{2.17}$$

where we have interchanged the order of summation and integration because of absolute convergence. But from [14, p. 115, formula 11.1], for $c=$ Re $s>\pm$ Re ν,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2^{s-2}a^{-s}\Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)x^{-s}\,ds=K_\nu(ax). \tag{2.18}$$

Hence,

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty} \Gamma^2\left(\frac{s}{2}\right)\left(\frac{\pi m}{y}\right)^{-s} ds = 8\pi i K_0\left(\frac{2\pi m}{y}\right). \tag{2.19}$$

Thus from (2.17) and (2.19), we conclude that

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty} \pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)y^s\, ds = 8\pi i\sum_{m=1}^{\infty} d(m)K_0\left(\frac{2\pi m}{y}\right). \tag{2.20}$$

Then from (2.5), (2.13), (2.16) and (2.20), we see that

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)y^s\, ds = -2\pi i\left(y\left(\gamma-\log\left(\frac{4\pi}{y}\right)\right)-4\sum_{m=1}^{\infty} d(m)K_0\left(\frac{2\pi m}{y}\right)\right). \tag{2.21}$$

Hence from (2.3) and (2.21), we deduce that

$$\int_0^{\infty} \frac{64\left(\Xi\left(\frac{t}{2}\right)\right)^2\cos nt}{(1+t^2)^2}\, dt = -\frac{2\pi}{\sqrt{y}}\left(y\left(\gamma-\log\left(\frac{4\pi}{y}\right)\right)-4\sum_{m=1}^{\infty} d(m)K_0\left(\frac{2\pi m}{y}\right)\right). \tag{2.22}$$

Let $n=\frac{1}{2}\log\alpha$ so that $y=\alpha$. Since $\alpha\beta=1$, we see that

$$\int_0^{\infty} \frac{64\left(\Xi\left(\frac{t}{2}\right)\right)^2\cos\left(\frac{1}{2}t\log\alpha\right)}{(1+t^2)^2}\, dt = -2\pi\sqrt{\beta}\left(\frac{\gamma-\log(4\pi\beta)}{\beta}-4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\beta)\right). \tag{2.23}$$

Switching α and β in (2.23) and then combining with (2.23) and simplifying, we arrive at (1.12), since the left-hand side of (2.23) is invariant under the map $\alpha\to\beta$.

Corollary 2.1. *Let $G(x)=\sum_{n=1}^{\infty} e^{-\pi n^2 x}$. Then,*

$$4\sum_{n=1}^{\infty} d(n)K_0(2\pi n) = \gamma-\log(4\pi)+\int_0^{\infty}\left(G(x^2)-1-\frac{1}{x}\right)^2 dx. \tag{2.24}$$

Proof. Letting $\alpha=1$ in (1.12), we see that

$$\gamma-\log(4\pi)-4\sum_{n=1}^{\infty} d(n)K_0(2\pi n) = -\frac{32}{\pi}\int_0^{\infty}\frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2}{(1+t^2)^2}\, dt. \tag{2.25}$$

Now from Theorem 4 in [12], we have for $0<\sigma<1$ that

$$\int_{-\infty}^{\infty}\left|\zeta(\sigma+it)\Gamma(\tfrac{1}{2}\sigma+\tfrac{1}{2}it)\right|^2\, dt = 2\pi^{1+\sigma}\int_0^{\infty}(uG(u^2)-1-u)^2u^{2\sigma-3}\, du. \tag{2.26}$$

Substituting $\sigma=1/2$ in (2.26) and writing the left-hand side in terms of the Riemann-Ξ function, we see that

$$\int_0^{\infty}\frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2}{(1+t^2)^2}\, dt = \frac{\pi}{32}\int_0^{\infty}\left(G(x^2)-1-\frac{1}{x}\right)^2 dx. \tag{2.27}$$

Combining (2.25) and (2.27), we obtain (2.24). □

Remark. Equation (2.24) is not unlike the first equality of a formula on page 254 in Ramanujan's Lost Notebook [20], given below in a different version in [3], i.e.,

$$\int_0^\infty \frac{dx}{x(e^{2\pi x}-1)(e^{2\pi a/x}-1)} = 2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{an}) \tag{2.28}$$

$$= \frac{a}{\pi^2}\sum_{n=1}^\infty \frac{d(n)\log(a/n)}{a^2-n^2} - \frac{1}{2}\gamma - \left(\frac{1}{4}+\frac{1}{4\pi^2 a}\right)\log a - \frac{\log(2\pi)}{2\pi^2 a}. \tag{2.29}$$

3. Another proof of Ramanujan's transformation formula (1.9)

In this section we give a new proof of Ramanujan's transformation formula (1.9) employing the same method used for proving Koshliakov's formula, i.e., by using (1.21). Let

$$f(t) := \frac{\Xi(t)}{\frac{1}{4}+t^2}\Gamma\left(-\frac{1}{4}+\frac{it}{2}\right)\Gamma\left(-\frac{1}{4}-\frac{it}{2}\right) \tag{3.1}$$

Then from (1.19), we deduce that

$$\phi(s) = \frac{\sqrt{\xi(\frac{1}{2}-s)}}{\frac{1}{2}+s}\Gamma\left(-\frac{1}{4}+\frac{s}{2}\right). \tag{3.2}$$

Thus from (1.21) with $y = e^{2n}$, we find that

$$\begin{aligned}
&\int_0^\infty 4\left(\Xi\left(\frac{t}{2}\right)\right)^2\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\frac{\cos nt}{1+t^2}\,dt \\
&= -\frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\xi^2(s)}{s(s-1)}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)y^s\,ds \\
&= -\frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{s(s-1)}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s ds,
\end{aligned} \tag{3.3}$$

where in the penultimate line, we have made use of (1.7). Now to examine the integral in the last expression in (3.3), we wish to move the line of integration from Re $s=\frac{1}{2}$ to Re $s = 1+\delta$, for some $\delta\in(0,1)$, so that we can use (1.2). But while doing that, we need to take care of the pole of order 2 at $s=1$ of the integrand in the last expression in (3.3).

Let $T>0$ denote a real number. Then by the residue theorem, we know that

$$\begin{aligned}
&\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\frac{s(s-1)}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s ds \\
&= \left[\int_{\frac{1}{2}-iT}^{1+\delta-iT}+\int_{1+\delta-iT}^{1+\delta+iT}+\int_{1+\delta+iT}^{\frac{1}{2}+iT}\right]\frac{s(s-1)}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s ds \\
&\quad - 2\pi i\lim_{s\to1}\frac{d}{ds}\left(\frac{s(s-1)^3}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s\right).
\end{aligned} \tag{3.4}$$

First we evaluate

$$\lim_{s\to 1}\frac{d}{ds}\left(\frac{s(s-1)^3}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s\right). \tag{3.5}$$

Using the product rule for differentiation and then simplifying, we see that

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{s(s-1)^3}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s\right)\\
&=-\left(\frac{d}{ds}\left((s-1)^2\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s\right)\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{2-s}{2}\right)\\
&\quad-\frac{1}{2}\left((s-1)^2\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s\right)\Gamma'\left(\frac{s+1}{2}\right)\Gamma\left(\frac{2-s}{2}\right)\\
&\quad+\frac{1}{2}\left((s-1)^2\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma'\left(\frac{2-s}{2}\right).
\end{aligned} \tag{3.6}$$

Then from (2.12), (2.13), (3.6) and the fact that $\frac{\Gamma'(1)}{\Gamma(1)}=-\gamma$, we deduce that

$$\lim_{s\to 1}\frac{d}{ds}\left(\frac{s(s-1)^3}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s\right)=-y\sqrt{\pi}\left(\gamma-\log\left(\frac{2\pi}{y}\right)\right). \tag{3.7}$$

From (2.14), we observe that,

$$\lim_{T\to\infty}\int_{\frac{1}{2}\pm iT}^{1+\delta\pm iT}\frac{s(s-1)}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s\,ds=0. \tag{3.8}$$

Lastly,

$$\begin{aligned}
&\int_{1+\delta-i\infty}^{1+\delta+i\infty}\frac{s(s-1)}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^s\,ds\\
&=\sum_{k=1}^{\infty}\int_{1+\delta-i\infty}^{1+\delta+i\infty}\frac{s(s-1)}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta(s)\left(\frac{k\pi}{y}\right)^{-s}\,ds,
\end{aligned} \tag{3.9}$$

where in the last step, we have interchanged the order of summation and integration because of absolute convergence. We simplify the integrand using Legendre's duplication formula [21, p. 46], namely,

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2s-1}}\Gamma(2s), \tag{3.10}$$

the reflection formula for Gamma function [21, p. 46]

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}, \tag{3.11}$$

for $s\notin\mathbb{Z}$ and the functional equation for the Riemann zeta function [22, p. 13, eqn. (2.1.1)],

$$\zeta(s)=2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\tfrac{1}{2}\pi s\right). \tag{3.12}$$

Thus,

$$\begin{aligned}&\frac{s(s-1)}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta(s)\left(\frac{k\pi}{y}\right)^{-s}\\&=-\frac{\pi^{3/2}}{2^{s-1}}\frac{\Gamma(s)\zeta(s)}{\sin\left(\frac{1}{2}\pi s\right)}\left(\frac{k\pi}{y}\right)^{-s}z\\&=-2\pi^{3/2}\frac{\zeta(1-s)}{\sin\pi s}\left(\frac{k}{y}\right)^{-s}.\end{aligned}\tag{3.13}$$

Hence from (3.9) and (3.13), we find that

$$\begin{aligned}&\int_{1+\delta-i\infty}^{1+\delta+i\infty}\frac{s(s-1)}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^{s}ds\\&=-2\pi^{3/2}\sum_{k=1}^{\infty}\int_{1+\delta-i\infty}^{1+\delta+i\infty}\frac{\zeta(1-s)}{\sin\pi s}\left(\frac{k}{y}\right)^{-s}ds.\end{aligned}\tag{3.14}$$

Now from [14, p. 201, formula 5.74], we know that if $0<\text{Re } s<1$, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(1-s)}{\sin\pi s}x^{-s}\,ds=-\frac{1}{\pi}\left(\psi(x+1)-\log x\right).\tag{3.15}$$

Hence we need to shift the line of integration from Re $s=1+\delta$ to Re $s=c$, where $c\in(0,1)$ and then use the residue theorem. While doing that, we encounter a pole of order 1 at $s=1$ of the integrand in the last expression in (3.14). Thus by another application of the residue theorem, we see that

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty}\frac{\zeta(1-s)}{\sin\pi s}\left(\frac{k}{y}\right)^{-s}ds=\int_{c-i\infty}^{c+i\infty}\frac{\zeta(1-s)}{\sin\pi s}\left(\frac{k}{y}\right)^{-s}ds+2\pi i\lim_{s\to1}\frac{(s-1)\zeta(1-s)}{\sin\pi s}\left(\frac{k}{y}\right)^{-s}\tag{3.16}$$

But

$$\lim_{s\to1}\frac{(s-1)\zeta(1-s)}{\sin\pi s}\left(\frac{k}{y}\right)^{-s}=\frac{y}{k}\lim_{s\to1}\frac{\zeta(1-s)-(s-1)\zeta'(1-s)}{\pi\cos\pi s}=\frac{y}{2k\pi},$$

since $\zeta(0)=-\frac{1}{2}$ and $\zeta'(0)=-\frac{1}{2}\log(2\pi)$ [22, pp. 19–20, eqns. (2.4.3), (2.4.5)]. Hence from (3.14), (3.15), (3.16), and (3.17), we deduce that

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty}\frac{s(s-1)}{4}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{-s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)\left(\frac{y}{\pi}\right)^{s}ds=4\pi^{\frac{3}{2}}i\sum_{k=1}^{\infty}\left(\psi\left(\frac{k}{y}\right)+\frac{y}{2k}-\log\left(\frac{k}{y}\right)\right),\tag{3.17}$$

since $\psi(x+1)=\psi(x)+1/x$ [21, p. 54]. Finally from (3.3), (3.4), (3.7), (3.8) and (3.17), we observe that

$$\begin{aligned}&-\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos nt}{1+t^2}\,dt\\&=\frac{1}{\sqrt{y}}\left(\sum_{k=1}^{\infty}\left(\psi\left(\frac{k}{y}\right)+\frac{1}{2k/y}-\log\left(\frac{k}{y}\right)\right)+\frac{y}{2}\left(\gamma-\log\left(\frac{2\pi}{y}\right)\right)\right).\end{aligned}\tag{3.18}$$

Letting $n = \frac{1}{2}\log\alpha$ in (3.18), where $\alpha\beta = 1$, and noting that $y = e^{2n} = \alpha$, we have

$$-\frac{1}{\pi^{3/2}}\int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt$$
$$= \sqrt{\beta}\left(\frac{\gamma-\log(2\pi\beta)}{2\beta} + \sum_{k=1}^{\infty}\left(\psi(k\beta) + \frac{1}{2k\beta} - \log(k\beta)\right)\right). \tag{3.19}$$

Now switching α and β in (3.19), combining with (3.19), and then using (1.5), we arrive at (1.9), since the left-hand side of (3.19) is invariant under the map $\alpha \to \beta$. This proves Ramanujan's transformation formula.

4. Ferrar's formula

In this section, we give a brief sketch of a proof of the extended version of Ferrar's formula (1.13). The steps in the latter part of the proof are similar to those given by Ferrar [6]; but we give them here for self-containedness.

Let

$$f(t) := \frac{2}{\frac{1}{4}+t^2}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}\right). \tag{4.1}$$

Then using (1.19), we see that

$$\phi(s) = \frac{\sqrt{2}}{\left(\frac{1}{2}-s\right)}\Gamma\left(\frac{1}{4}+\frac{s}{2}\right). \tag{4.2}$$

Hence from (1.21) with $y = e^{2n}$, we find that

$$8\int_0^\infty \Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\frac{\cos nt}{1+t^2}\,dt = -\frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\pi^{-\frac{s}{2}}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)y^s\,ds. \tag{4.3}$$

While examining the integral in the last expression in (4.3), we shift the line of integration from Re $s = \frac{1}{2}$ to Re $s = 1+\delta$, for some $\delta \in (0, 2)$, so that we can use (1.2). But while doing that, we need to take care of the pole of order 2 at $s = 1$ of the integrand in the last expression in (4.3).

Let $T > 0$ denote a real number. Then by the residue theorem, we know that

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\pi^{-\frac{s}{2}}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)y^s\,ds$$
$$= \left[\int_{\frac{1}{2}-iT}^{1+\delta-iT} + \int_{1+\delta-iT}^{1+\delta+iT} + \int_{1+\delta+iT}^{\frac{1}{2}+iT}\right]\pi^{-\frac{s}{2}}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)y^s\,ds$$
$$-2\pi i\lim_{s\to1}\frac{d}{ds}\left((s-1)^2\pi^{-\frac{s}{2}}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)y^s\right). \tag{4.4}$$

Using the product rule for differentiation and simplifying, we have

$$\begin{aligned}
&\frac{d}{ds}\left((s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^s\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)\right)\\
&=(s-1)\left(\frac{y}{\sqrt{\pi}}\right)^s\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(2\zeta(s)-\frac{1}{2}(s-1)\psi\left(\frac{1-s}{2}\right)\zeta(s)+(s-1)\zeta'(s)\right)\\
&\quad+(s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^s\log\left(\frac{y}{\sqrt{\pi}}\right)\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)\\
&\quad+(s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^s\Gamma^2\left(\frac{s}{2}\right)\psi\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)\\
&=f_1(s)+f_2(s)+f_3(s),
\end{aligned}\tag{4.5}$$

say.

Now we need the well-known Laurent expansion [7, p. 944, formula 8.321, no. 1]

$$\Gamma(s)=\frac{1}{s}-\gamma+\cdots,\tag{4.6}$$

so that

$$\psi\left(\frac{1-s}{2}\right)=\frac{2}{s-1}-\gamma+\cdots.\tag{4.7}$$

Then from (2.7) and (4.7), we have

$$\lim_{s\to1}\left(2\zeta(s)-\frac{1}{2}(s-1)\psi\left(\frac{1-s}{2}\right)\zeta(s)+(s-1)\zeta'(s)\right)=\frac{3\gamma}{2},\tag{4.8}$$

so that from (4.5) and (4.8), we obtain

$$\begin{aligned}
\lim_{s\to1}f_1(s)&=\lim_{s\to1}(s-1)\left(\frac{y}{\sqrt{\pi}}\right)^s\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(2\zeta(s)-\frac{1}{2}(s-1)\psi\left(\frac{1-s}{2}\right)\zeta(s)+(s-1)\zeta'(s)\right)\\
&=-3y\gamma\sqrt{\pi}.
\end{aligned}\tag{4.9}$$

Also,

$$\lim_{s\to1}f_2(s)=\lim_{s\to1}(s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^s\log\left(\frac{y}{\sqrt{\pi}}\right)\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)=-2y\sqrt{\pi}\log\left(\frac{y}{\sqrt{\pi}}\right),\tag{4.10}$$

and using (2.12), we find that

$$\lim_{s\to1}f_3(s)=\lim_{s\to1}(s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^s\Gamma^2\left(\frac{s}{2}\right)\psi\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)=-2y\sqrt{\pi}\left(-\gamma-2\log 2\right).\tag{4.11}$$

Finally from (4.9), (4.10) and (4.11), we deduce that

$$\lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^{-s}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)\right)$$
$$=-3y\gamma\sqrt{\pi}-2y\sqrt{\pi}\log\left(\frac{y}{\sqrt{\pi}}\right)-2y\sqrt{\pi}\left(-\gamma-2\log 2\right)$$
$$=y\sqrt{\pi}\left(\log 16\pi-2\log y-\gamma\right). \tag{4.12}$$

Let $T\to\infty$ in (4.4). Then the integrals along the horizontal segments $[\frac{1}{2}-iT, 1+\delta-iT]$ and $[1+\delta+iT, \frac{1}{2}+iT]$ tend to 0. Next using (1.2), we have

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty}\pi^{-\frac{s}{2}}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)y^s\,ds=\sum_{m=1}^{\infty}\int_{1+\delta-i\infty}^{1+\delta+i\infty}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}ds, \tag{4.13}$$

where we have interchanged the order of summation and integration because of absolute convergence. Now from [14, p. 115, formula 11.4], we know that if $\pm$ Re $\nu<c=$ Re $s<1/2$, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\pi^{-\frac{1}{2}}(2a)^{-s}\cos(\pi\nu)\Gamma\left(\tfrac{1}{2}-s\right)\Gamma(s+\nu)\Gamma(s-\nu)x^{-s}\,ds=e^{ax}K_{\nu}(ax). \tag{4.14}$$

Letting $\nu=0$, and replacing s by $s/2$ and a by $a/2$ in (4.14), we find that for $0<c=$ Re $s<1$,

$$\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty}a^{-\frac{s}{2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)x^{-\frac{s}{2}}\,ds=\pi^{\frac{1}{2}}e^{\frac{1}{2}ax}K_0\left(\frac{1}{2}ax\right). \tag{4.15}$$

Now let $a=1$ and $x=\pi m^2/y^2$ in (4.15). Then for $0<c=$ Re $s<1$, we have

$$\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{1-s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}ds=4\pi^{\frac{3}{2}}ie^{\frac{\pi m^2}{2y^2}}K_0\left(\frac{\pi m^2}{2y^2}\right). \tag{4.16}$$

But since (4.16) is valid only for $0<$ Re $s<1$, in order to simplify the integral on the right-hand side of (4.13), we need to shift the line of integration from Re $s=1+\delta$, where $\delta\in(0,2)$ to Re $s=c$, where $c\in(0,1)$ and then use the residue theorem. While doing that, we encounter a pole of order 1 at $s=1$ of the integrand on the right-hand side of (4.13). Thus by another application of the residue theorem, we see that

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}ds$$
$$=\int_{c-i\infty}^{c+i\infty}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}ds+2\pi i\lim_{s\to 1}(s-1)\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s} \tag{4.17}$$

First,

$$\lim_{s\to 1}(s-1)\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}=-\frac{2\sqrt{\pi}}{m/y}. \tag{4.18}$$

Hence from (4.16), (4.17) and (4.18), we find that

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty} \Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s} ds = 2\pi i\left(2\sqrt{\pi}\sum_{m=1}^{\infty}\left(e^{\frac{\pi m^2}{2y^2}}K_0\left(\frac{\pi m^2}{2y^2}\right)-\frac{1}{m/y}\right)\right). \tag{4.19}$$

Then from (4.3), (4.4), (4.12) and (4.19), we see that

$$\begin{aligned}
&8\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\frac{\cos nt}{1+t^2}\,dt\\
&=\frac{-1}{i\sqrt{y}}\left(2\pi i\left(2\sqrt{\pi}\sum_{m=1}^{\infty}\left(e^{\frac{\pi m^2}{2y^2}}K_0\left(\frac{\pi m^2}{2y^2}\right)-\frac{1}{m/y}\right)-y\sqrt{\pi}\left(\log 16\pi-2\log y-\gamma\right)\right)\right)\\
&=\frac{2\pi^{3/2}}{\sqrt{y}}\left(y\left(\log 16\pi-2\log y-\gamma\right)-2\sum_{m=1}^{\infty}\left(e^{\frac{\pi m^2}{2y^2}}K_0\left(\frac{\pi m^2}{2y^2}\right)-\frac{1}{m/y}\right)\right).
\end{aligned} \tag{4.20}$$

Now letting $n=\frac{1}{2}\log\alpha$ in (4.20), noting that $y=e^{2n}$ and $\alpha\beta=1$ and then simplifying, we observe that

$$\begin{aligned}
&4\pi^{-3/2}\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt\\
&=\sqrt{\beta}\left(\frac{\log 16\pi+2\log\beta-\gamma}{\beta}-2\sum_{m=1}^{\infty}\left(e^{\frac{\pi m^2\beta^2}{2}}K_0\left(\frac{\pi m^2\beta^2}{2}\right)-\frac{1}{m\beta}\right)\right).
\end{aligned} \tag{4.21}$$

Now switching α and β in (4.21) and combining with (4.21), we arrive at (1.13), since the left-hand side of (4.21) is invariant under the map $\alpha\to\beta$.

5. Hardy's formula

Here we give a brief sketch of a proof of Hardy's formula (1.17). Let

$$f(t):=\frac{1}{32\pi^2}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}-\frac{it}{2}\right). \tag{5.1}$$

Using (1.19), we observe that

$$\phi(s)=\frac{1}{4\sqrt{2}\pi}\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{-1}{4}+\frac{s}{2}\right), \tag{5.2}$$

and using (3.11), we find that

$$\begin{aligned}
f\left(\frac{t}{2}\right)&=\frac{1}{32\pi^2}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\\
&=\frac{1}{1+t^2}\frac{1}{2\sin\left(\pi\left(\frac{1+it}{4}\right)\right)\sin\left(\pi\left(\frac{1-it}{4}\right)\right)}\\
&=\frac{1}{1+t^2}\frac{1}{\cos\left(\frac{i\pi t}{2}\right)-\cos\left(\frac{\pi}{2}\right)}\\
&=\frac{1}{(1+t^2)\cosh\frac{1}{2}\pi t}.
\end{aligned} \tag{5.3}$$

Hence from (1.21) with $y = e^{2n}$, (3.10) and (1.7), we see that

$$\int_0^\infty \frac{\Xi(\frac{1}{2}t)}{1+t^2}\frac{\cos nt}{\cosh\frac{1}{2}\pi t}\,dt = \frac{1}{32\pi^2 i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{-s}{2}\right)\xi(s)y^s\,ds$$
$$= \frac{1}{4\pi i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s-1)\Gamma(-s)(s-1)\Gamma\left(1+\frac{s}{2}\right)\pi^{-s/2}\zeta(s)y^s\,ds, \tag{5.4}$$

To examine the integral in the last expression in (5.4), we wish to move the line of integration from Re $s = \frac{1}{2}$ to Re $s = 1+\delta$, for some $\delta \in (0,1)$, so that we can use (1.2). In this process, we encounter the pole of order 2 of the integrand at $s = 1$ in the last expression in (5.4).

Let $T > 0$ denote a real number. Then by the residue theorem, we know that

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\Gamma(s-1)\Gamma(-s)(s-1)\Gamma\left(1+\frac{s}{2}\right)\pi^{-s/2}\zeta(s)y^s\,ds$$
$$= \left[\int_{\frac{1}{2}-iT}^{1+\delta-iT}+\int_{1+\delta-iT}^{1+\delta+iT}+\int_{1+\delta+iT}^{\frac{1}{2}+iT}\right]\Gamma(s-1)\Gamma(-s)(s-1)\Gamma\left(1+\frac{s}{2}\right)\pi^{-s/2}\zeta(s)y^s\,ds$$
$$-2\pi i\lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\Gamma(s-1)\Gamma(-s)\xi(s)y^s\right). \tag{5.5}$$

Now

$$\frac{d}{ds}\left((s-1)^2\Gamma(s-1)\Gamma(-s)\xi(s)y^s\right)$$
$$= \left(\frac{d}{ds}\left((s-1)^2\Gamma(s-1)\Gamma(-s)\right)\right)\xi(s)y^s + \left((s-1)^2\Gamma(s-1)\Gamma(-s)\xi^{'}(s)y^s\right)$$
$$+(s-1)^2\Gamma(s-1)\Gamma(-s)\xi(s)y^s\log y. \tag{5.6}$$

Using (3.11) to simplify the first expression on the right-hand side of (5.6) and then using L'Hopital's rule twice, we easily see that

$$\lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\Gamma(s-1)\Gamma(-s)\right) = -1. \tag{5.7}$$

Now $\Gamma(s)$ has a residue $\frac{(-1)^n}{n!}$ at $s = -n$ where n is a positive integer [7, p. 883, formula 8.310, no. 2]. Hence we see that

$$\lim_{s\to 1}(s-1)\Gamma(-s) = 1. \tag{5.8}$$

Also from [4, pp. 80–81], we know that

$$\xi(1) = \frac{1}{2}, \tag{5.9}$$

and

$$-\frac{\xi^{'}(1)}{\xi(1)} = -\frac{\gamma}{2} - 1 + \frac{1}{2}\log 4\pi, \tag{5.10}$$

so that

$$\xi^{'}(1)=\frac{\gamma}{4}+\frac{1}{2}-\frac{1}{4}\log 4\pi. \tag{5.11}$$

Thus from (5.6), (5.7), (5.8), (5.9) and (5.11), we deduce that

$$\lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\Gamma(s-1)\Gamma(-s)\xi(s)y^s\right)=\frac{y}{4}\left(\gamma-\log 4\pi+2\log y\right). \tag{5.12}$$

Now let $T\to\infty$ in (5.5). The integrals along the horizontal segments $[\frac{1}{2}-iT,1+\delta-iT]$ and $[1+\delta+iT,\frac{1}{2}+iT]$ tend to 0. Finally,

$$\begin{aligned}
&\int_{1+\delta-i\infty}^{1+\delta+i\infty}\Gamma(s-1)\Gamma(-s)(s-1)\Gamma\left(1+\frac{s}{2}\right)\pi^{-s/2}\zeta(s)y^s\,ds\\
&=\int_{1+\delta-i\infty}^{1+\delta+i\infty}-\frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s)y^s\,ds\\
&=\int_{1+\delta-i\infty}^{1+\delta+i\infty}-\frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\sum_{k=1}^{\infty}\frac{1}{k^s}y^s\,ds\\
&=\sum_{k=1}^{\infty}\int_{1+\delta-i\infty}^{1+\delta+i\infty}-\frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s}{2}\right)\left(\frac{k\sqrt{\pi}}{y}\right)^{-s}ds,
\end{aligned} \tag{5.13}$$

where in the last step, we have interchanged the order of summation and integration because of absolute convergence. Now employing a change of variable $s\to s+1$, we see that

$$\sum_{k=1}^{\infty}\int_{1+\delta-i\infty}^{1+\delta+i\infty}-\frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s}{2}\right)\left(\frac{k\sqrt{\pi}}{y}\right)^{-s}ds=\sum_{k=1}^{\infty}\frac{y}{k\sqrt{\pi}}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s+1}{2}\right)\left(\frac{k\sqrt{\pi}}{y}\right)^{-s}ds. \tag{5.14}$$

Next, let $F(s)$ denote the Mellin transform of $f(x)$, i.e.,

$$\mathcal{M}(f(x);s)=F(s):=\int_0^{\infty}x^{s-1}f(x)\,dx. \tag{5.15}$$

Then the inverse Mellin transform is given by [22, p. 33]

$$\mathcal{M}^{-1}(F(s);x)=f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)x^{-s}\,ds, \tag{5.16}$$

where $c=\text{Re}\ s$ lies in the fundamental strip (or the strip of analyticity) for which $F(s)$ is defined. Since [15, p. 404]

$$\mathcal{M}^{-1}\left(F(s+a);x\right)=x^af(x), \tag{5.17}$$

and [15, p.406]

$$\mathcal{M}^{-1}\left(\frac{1}{2}\Gamma\left(\frac{s}{2}\right);x\right)=e^{-x^2}, \tag{5.18}$$

for $\text{Re}\ s>0$, we see that

$$\mathcal{M}^{-1}\left(\frac{1}{2}\Gamma\left(\frac{s+1}{2}\right);x\right)=xe^{-x^2}. \tag{5.19}$$

Also for $0<\text{Re } s<1$, we have [15, p. 91, eqn. (3.3.10)]

$$\mathcal{M}^{-1}\left(\frac{\pi}{\sin\pi s};x\right)=\frac{1}{1+x}. \tag{5.20}$$

But from [15, p.83, eqn. (3.1.13)], we observe that

$$\mathcal{M}^{-1}\left(F(s)G(s);w\right)=\int_0^{\infty}f(x)g\left(\frac{w}{x}\right)\frac{dx}{x}, \tag{5.21}$$

where $F(s)$ and $G(s)$ are Mellin transforms of $f(x)$ and $g(x)$ respectively.

Thus from (1.6), (5.19), (5.20) and (5.21), we see that for $0<\delta<1$,

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{y}{k\sqrt{\pi}}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s+1}{2}\right)\left(\frac{k\sqrt{\pi}}{y}\right)^{-s}ds\\
&=2\pi i\sum_{k=1}^{\infty}\frac{y}{k\sqrt{\pi}}\int_0^{\infty}\frac{xe^{-x^2}}{1+\frac{k\sqrt{\pi}}{xy}}\frac{dx}{x}\\
&=2\pi i\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty}\frac{xe^{-\pi x^2y^{-2}}}{x+k}\,dx\\
&=2\pi i\int_0^{\infty}e^{-\pi x^2y^{-2}}\sum_{k=1}^{\infty}\frac{x}{k(x+k)}\,dx\\
&=2\pi i\int_0^{\infty}e^{-\pi x^2y^{-2}}\sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{x+1+k}\right)dx\\
&=2\pi i\int_0^{\infty}e^{-\pi x^2y^{-2}}\left(\psi(x+1)+\gamma\right)dx,\\
&=2\pi i\left(\frac{\gamma y}{2}+\int_0^{\infty}\psi(x+1)e^{-\pi x^2y^{-2}}\,dx\right),
\end{aligned} \tag{5.22}$$

since $\int_0^{\infty}e^{-ax^2}\,dx=\frac{\sqrt{\pi}}{2\sqrt{a}}$. Here again the interchange of the order of integration and summation is justified by absolute convergence. Thus from (5.4), (5.5), (5.12), (5.13), (5.14) and (5.22), we deduce that

$$\begin{aligned}
\int_0^{\infty}\frac{\Xi(\frac{1}{2}t)}{1+t^2}\frac{\cos nt}{\cosh\frac{1}{2}\pi t}\,dt&=\frac{1}{4\pi i\sqrt{y}}\left(2\pi i\left(\frac{\gamma y}{2}+\int_0^{\infty}\psi(x+1)e^{-\pi x^2y^{-2}}\,dx\right)-\frac{2\pi iy}{4}\left(\gamma-\log 4\pi+2\log y\right)\right)\\
&=\frac{e^n}{4}\left(-2n+\frac{1}{2}\gamma+\frac{1}{2}\log\pi+\log 2\right)+\frac{e^{-n}}{2}\int_0^{\infty}\psi(x+1)e^{-\pi x^2e^{-4n}}\,dx.
\end{aligned} \tag{5.23}$$

Finally, since the left-hand side of (5.23) is an even function of n, replacing n by $-n$ in (5.23), we obtain (1.17). This completes the proof.

Acknowledgements

The author wishes to express his gratitude to Professor Bruce C. Berndt for his constant support, careful reading of this manuscript and for helpful comments.

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