TRANSFORMATION FORMULAS ASSOCIATED WITH INTEGRALS INVOLVING THE RIEMANN Ξ-FUNCTION

ATUL DIXIT

Abstract. Using residue calculus and the theory of Mellin transforms, we evaluate integrals of a certain type involving the Riemann Ξ-function, which give transformation formulas of the form $F(z,\alpha)=F(z,\beta)$, where $\alpha\beta=1$. This gives a unified approach for generating modular transformation formulas, including a famous formula of Ramanujan and Guinand.

1. Introduction

Let Riemann's ξ-function be defined by

$$\xi(s):=(s-1)\pi^{-\frac{1}{2}s}\Gamma(1+\tfrac{1}{2}s)\zeta(s), \tag{1.1}$$

and let

$$\Xi(t):=\xi(\tfrac{1}{2}+it) \tag{1.2}$$

be the Riemann Ξ-function. In [11], S. Ramanujan introduced the integral

$$\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt, \tag{1.3}$$

where Re z is not an integer. There he gave alternative representations for this integral when Re $z>1$, $-1<$ Re $z<1$, $-3<$ Re $z<-1$ and so on. It turns out, as we show here, that (1.3) gives rise to nice transformation formulas involving the Hurwitz zeta function that are modular in nature, one of which is a generalization of the following beautiful transformation formula of Ramanujan (see [10, p. 220]) recently proved in [1].

Theorem 1.1. *Define*

$$\lambda(x):=\psi(x)+\frac{1}{2x}-\log x, \tag{1.4}$$

where

$$\psi(x):=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma-\sum_{m=0}^{\infty}\left(\frac{1}{m+x}-\frac{1}{m+1}\right), \tag{1.5}$$

the logarithmic derivative of Gamma function. Let the Riemann's ξ and Ξ functions be defined as in (1.1) *and* (1.2) *respectively. If α and β are positive numbers such that*

$\alpha\beta = 1$, *then*

$$\sqrt{\alpha}\left\{\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{n=1}^{\infty}\lambda(n\alpha)\right\}=\sqrt{\beta}\left\{\frac{\gamma-\log(2\pi\beta)}{2\beta}+\sum_{n=1}^{\infty}\lambda(n\beta)\right\}$$
$$=-\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt, \quad (1.6)$$

where γ denotes Euler's constant.

Here we study a general integral of the form

$$\int_0^{\infty} f\left(z,\frac{t}{2}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\cos nt\,dt, \quad (1.7)$$

where $f(z,t)=\phi(z,it)\phi(z,-it)$ and n is real. This includes (1.3) as a special case with

$$f(z,t)=\frac{1}{(4t^2+(z+1)^2)}\Gamma\left(\frac{z-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{z-1}{4}-\frac{it}{2}\right). \quad (1.8)$$

This integral always gives rise to modular transformation formulas of the form $F(z,\alpha)=F(z,\beta)$, where $\alpha\beta=1$. Letting $z\to 0$ in these formulas then gives transformation formulas of the form $F(\alpha)=F(\beta)$ for $\alpha\beta=1$, where the integral involving Riemann Ξ-function giving rise to these formulas is of the form

$$\int_0^{\infty} f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)\cos nt\,dt, \quad (1.9)$$

with $f(t)=\phi(it)\phi(-it)$. An example of such a formula is (1.6), which stems from the integral in (1.9) with

$$f(t):=\frac{\Xi(t)}{\frac{1}{4}+t^2}\Gamma\left(-\frac{1}{4}+\frac{it}{2}\right)\Gamma\left(-\frac{1}{4}-\frac{it}{2}\right). \quad (1.10)$$

In [12, p. 35], a small section is devoted to the integral in (1.9). For more details, see [5].

Another special case of (1.7) that we study here extends Guinand's formula [8], which, in fact, was discovered by Ramanujan several years earlier (see [2, p. 253]). This integral does not appear to have been studied before. Ramanujan's version of Guinand's formula is given below.

Theorem 1.2 (Ramanujan-Guinand formula)**.** *Let $K_\nu(z)$ denote the modified Bessel function of order ν and let $\sigma_k(n)=\sum_{d|n}d^k$, let $\zeta(s)$ denote the Riemann zeta function. If α and β are positive numbers such that $\alpha\beta=\pi^2$, and if s is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{z/2}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{z/2}(2n\beta)$$
$$=\frac{1}{4}\Gamma\left(\frac{z}{2}\right)\zeta(z)\{\beta^{(1-z)/2}-\alpha^{(1-z)/2}\}+\frac{1}{4}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\{\beta^{(1+z)/2}-\alpha^{(1+z)/2}\}. \quad (1.11)$$

This identity is related to the Fourier expansion of nonanalytic Eisenstein series on $SL(2,\mathbb{Z})$, or Maass wave forms. See [2] for details. The proof of this identity in [2] as well as in [8] makes use of a theorem of Watson [13] given below.

Theorem 1.3. *Let $K_\nu(z)$ be defined as before. If $x>0$ and $\text{Re}\ \nu>0$, then*

$$\frac{1}{4}(\pi x)^{-\nu}\Gamma(\nu)+\sum_{n=1}^{\infty}n^{\nu}K_\nu(2\pi nx)$$
$$=\frac{1}{4}\sqrt{\pi}(\pi x)^{-\nu-1}\Gamma(\nu+\tfrac{1}{2})+\frac{\sqrt{\pi}}{2x}\left(\frac{x}{\pi}\right)^{\nu+1}\Gamma(\nu+\tfrac{1}{2})\sum_{n=1}^{\infty}(n^2+x^2)^{-\nu-1/2}. \quad (1.12)$$

Our extended version of the Ramanujan-Guinand formula given below not only gives a new proof of (1.11), which does not make use of Watson's theorem, but also shows a surprising connection between the Fourier expansion of Maass waveforms and the Riemann Ξ-function.

Theorem 1.4 (Extended version of Ramanujan-Guinand formula)**.** *Let $K_\nu(s),\sigma_k(n)$, and $\Xi(t)$ be defined as before and let $-1<\text{Re}\ z<1$. Then if α and β are positive numbers such that $\alpha\beta=1$, we have*

$$\sqrt{\alpha}\left(\alpha^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\alpha^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\alpha\right)\right)$$
$$=\sqrt{\beta}\left(\beta^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\beta^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\beta\right)\right)$$
$$=-\frac{32}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt. \quad (1.13)$$

The Hurwitz zeta function is defined for $\text{Re}\ z>1$ by

$$\zeta(z,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^z}. \quad (1.14)$$

It can also be analytically continued to the entire z-plane except for a simple pole at $z=1$. Our theorem involving the Hurwitz zeta function, which generalizes (1.6), is given below. A different proof of this theorem using the theory of special functions was obtained in [3].

Theorem 1.5. *Let $-1<\text{Re}\ z<1$. Define $\varphi(z,x)$ by*

$$\varphi(z,x)=\zeta(z+1,x)-\frac{x^{-z}}{z}-\frac{1}{2}x^{-z-1}, \quad (1.15)$$

where $\zeta(z,x)$ *denotes the Hurwitz zeta function. Then if* α *and* β *are any positive numbers such that* $\alpha\beta=1$,

$$\alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\varphi(z,n\alpha)-\frac{\zeta(z+1)}{2\alpha^{z+1}}-\frac{\zeta(z)}{\alpha z}\right)=\beta^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\varphi(z,n\beta)-\frac{\zeta(z+1)}{2\beta^{z+1}}-\frac{\zeta(z)}{\beta z}\right)$$
$$=\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt, \tag{1.16}$$

where $\Xi(t)$ *is defined in* (1.2).

Another formula involving the Hurwitz zeta function was introduced in [3]. It is as follows:

Theorem 1.6. *Let* Re $z>1$ *and let* $\zeta(z,a)$ *be the Hurwitz zeta function defined in* (1.14)*. If* α *and* β *are positive numbers such that* $\alpha\beta=1$*, then*

$$\alpha^{-\frac{(z+1)}{2}}\sum_{k=1}^{\infty}\zeta\left(z+1,1+\frac{k}{\alpha}\right)=\beta^{-\frac{(z+1)}{2}}\sum_{k=1}^{\infty}\zeta\left(z+1,1+\frac{k}{\beta}\right)$$
$$=\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt, \tag{1.17}$$

where Ξ *is defined as in* (1.2).

Corresponding to another strip $-3<$ Re $z<-1$, we get a different transformation formula as follows.

Theorem 1.7. *Let* $-3<$ Re $z<-1$. *Define* $\Lambda(z,x)$ *by*

$$\Lambda(z,x)=\zeta(z+1,x)-\frac{x^{-z}}{z}-\frac{1}{2}x^{-z-1}-\frac{(z+1)x^{-z-2}}{12}, \tag{1.18}$$

where $\zeta(z,x)$ *denotes the Hurwitz zeta function. Then if* α *and* β *are any positive numbers such that* $\alpha\beta=1$,

$$\alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\Lambda(z,n\alpha)-\frac{\zeta(z)}{\alpha z}+\frac{\zeta(z+1)}{2}+\frac{(z+1)\zeta(z+2)}{12\alpha^{z+2}}\right)$$
$$=\beta^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\Lambda(z,n\beta)-\frac{\zeta(z)}{\beta z}+\frac{\zeta(z+1)}{2}+\frac{(z+1)\zeta(z+2)}{12\beta^{z+2}}\right)$$
$$=\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt, \tag{1.19}$$

where $\Xi(t)$ *is defined in* (1.2).

The asymptotic expansion of $\zeta(z+1,x)$ for large $|x|$ and $|\arg x|<\pi$ is as follows [9, p. 25]:

$$\zeta(z+1,x)=\frac{1}{\Gamma(z+1)}\left(x^{-z}\Gamma(z)+\frac{1}{2}\Gamma(z+1)x^{-z-1}+\sum_{n=1}^{m-1}\frac{B_{2n}}{(2n)!}\Gamma(z+2n)x^{-2n-z}\right)+O(x^{-2m-z-2}). \tag{1.20}$$

As can be easily gleaned from (1.15), the definition of $\varphi(z,x)$, we have subtracted from $\zeta(z+1,x)$, the first two terms in its asymptotic expansion, whereas in (1.18), the definition of $\Lambda(z,x)$, we have subtracted from $\zeta(z+1,x)$, the first three terms in its asymptotic expansion. One can then construct a general transformation formula in the strip $-(2k+1)<\text{Re } z<-(2k-1)$, where $k\geq 0$, by subtracting from $\zeta(z+1,x)$, requisite number of terms for that particular strip from the asymptotic expansion (1.20). However, since the formulas soon start becoming complicated, we avoid providing details.

This paper is organized as follows. In Section 2, we give a complex integral representation of (1.7) that we subsequently use. Then in Section 3, we prove Theorem 1.4 and give its companion theorem when $|\text{Re } z|>1$. Finally in Section 4, we prove Theorem 1.5 and briefly sketch the proofs of Theorems 1.6 and 1.7 since they are along the similar lines as that of Theorem 1.5. All of these transformation formulas are proved using residue calculus and the theory of Mellin transforms, thus delineating a systematic approach for generating them. In the sequel, we use

$$R_a=R_a(g) \tag{1.21}$$

to denote the residue of a function g at a. We choose to write just R_a instead of $R_a(g)$ when the function in consideration is understood and there is not any ambiguity.

2. A complex integral representation of (1.7)

In this section, we give a formal calculation of the transformation of the integral in (1.7) into an equivalent complex integral which allows us to use residue calculus and Mellin transform techniques for its evaluation. This simple result is a generalization of a result given in [12, p. 35], where a complex integral representation was obtained for (1.9). This latter representation is utilized in [5].

Theorem 2.1. *Let*

$$f(z,t)=\phi(z,it)\phi(z,-it), \tag{2.1}$$

where ϕ is analytic in t as a function of a real variable and analytic in z in some domain. Assume that the integral in (1.7) *converges. Also let $y=e^n$ for n real. Then,*

$$\int_0^\infty f(z,t)\,\Xi\left(t+\frac{iz}{2}\right)\Xi\left(t+\frac{iz}{2}\right)\cos nt\,dt$$
$$=\frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\left(z,s-\frac{1}{2}\right)\phi\left(z,\frac{1}{2}-s\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)y^s\,ds. \tag{2.2}$$

Proof. Let

$$I(z,n) := \int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2}\right)\Xi\left(t-\frac{iz}{2}\right)\cos nt\,dt. \tag{2.3}$$

Then,

$$\begin{aligned}I(z,n) = \frac{1}{2}\bigg[&\int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2}\right)\Xi\left(t-\frac{iz}{2}\right)y^{it}\,dt\\&+\int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2}\right)\Xi\left(t-\frac{iz}{2}\right)y^{-it}\,dt\bigg].\end{aligned} \tag{2.4}$$

Now since $\xi(s)=\xi(1-s)$, where ξ is defined in (1.1), we have

$$\begin{aligned}&\Xi\left(\pm t+\frac{iz}{2}\right) = \xi\left(\frac{1}{2}\pm it-\frac{z}{2}\right) = \xi\left(1-\left(\frac{z+1}{2}\mp it\right)\right) = \xi\left(\frac{z+1}{2}\mp it\right),\\&\Xi\left(\pm t-\frac{iz}{2}\right) = \xi\left(\frac{1}{2}\pm it+\frac{z}{2}\right) = \xi\left(\frac{z+1}{2}\pm it\right).\end{aligned} \tag{2.5}$$

Hence,

$$\Xi\left(-t\pm\frac{iz}{2}\right) = \Xi\left(t\mp\frac{iz}{2}\right). \tag{2.6}$$

From (2.4), (2.6) and the fact that f is an even function of t, we deduce that

$$\begin{aligned}I(z,n) &= \frac{1}{2}\int_{-\infty}^\infty f(z,t)\Xi\left(t+\frac{iz}{2}\right)\Xi\left(t-\frac{iz}{2}\right)y^{it}\,dt\\&= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\left(z,s-\frac{1}{2}\right)\phi\left(z,\frac{1}{2}-s\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)y^s\,ds,\end{aligned} \tag{2.7}$$

where in the penultimate line, we made the change of variable $s=\frac{1}{2}+it$. □

For our purpose here, we replace n by $2n$ in (2.2) and then t by $t/2$ on the left-hand side of (2.2). Thus with $y=e^{2n}$, we find that

$$\begin{aligned}&\int_0^\infty f\left(z,\frac{t}{2}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\cos nt\,dt\\&= \frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\left(z,s-\frac{1}{2}\right)\phi\left(z,\frac{1}{2}-s\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)y^s\,ds.\end{aligned} \tag{2.8}$$

It is this equation with which we will be working throughout this paper.

3. Proof of Theorem 1.4

Let

$$f(z,t) = \frac{4}{\left(t^2+\frac{(z-1)^2}{4}\right)\left(t^2+\frac{(z+1)^2}{4}\right)}. \tag{3.1}$$

Substituting this representation of f in the integral in (1.7) gives (1.13). Using Stirling's formula on a vertical strip, which states that if $s = \sigma + it$, then for $\alpha \le \sigma \le \beta$ and $|t| \ge 1$,

$$|\Gamma(s)| = (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|}\left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{3.2}$$

as $t \to \infty$, one can show the convergence of the integral.

From (3.1) and (2.1), it can be easily seen that $\phi(z, s) = \frac{2}{\left(\frac{z-1}{2}-s\right)\left(\frac{z+1}{2}-s\right)}$. Thus using (2.8) with these f and ϕ, we find that

$$\begin{aligned}
&64\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos nt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt\\
&= \frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{4}{\left(\frac{z}{2}-s\right)\left(\frac{z+2}{2}-s\right)\left(\frac{z-2}{2}+s\right)\left(\frac{z}{2}+s\right)}\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)y^s\,ds\\
&= \frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds,
\end{aligned} \tag{3.3}$$

where in the penultimate line, we have used (1.1). To evaluate the integral in the last step in (3.3), we want to use the series representation for $\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)$, namely,

$$\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right) = \sum_{m=1}^\infty \frac{\sigma_{-z}(m)}{m^{s-\frac{z}{2}}}, \tag{3.4}$$

valid for Re $\left(s-\frac{z}{2}\right) > 1$ and Re $\left(s+\frac{z}{2}\right) > 1$, i.e., Re $s >$ Re $\left(\frac{z+2}{2}\right)$ and Re $s >$ Re $\left(\frac{2-z}{2}\right)$ (see [12, p. 8, eqn. 1.3.1]). But since $-1 <$ Re $z < 1$, we have $0 <$ Re $\left(s-\frac{z}{2}\right) < 1$ and $0 <$ Re $\left(s+\frac{z}{2}\right) < 1$. Thus we move the line of integration from Re $s = \frac{1}{2}$ to Re $s = \frac{3}{2}$. In this process, we encounter in the integrand a pole of order 1 at $s = \frac{z+2}{2}$ (due to $\zeta(s-\frac{z}{2})$) and a pole of order 1 at $s = \frac{2-z}{2}$ (due to $\zeta(s+\frac{z}{2})$).

Let $T > 0$ denote a real number. Then by the residue theorem and using the notation in (1.21), we know that

$$\begin{aligned}
&\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds\\
&= \left[\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}\right]\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds\\
&\quad - 2\pi i\left(R_{\frac{z+2}{2}} + R_{\frac{2-z}{2}}\right).
\end{aligned} \tag{3.5}$$

Now the functional equation for the Riemann zeta function [12, p. 22, eqn. (2.6.4)] gives,

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{(1-z)}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z). \tag{3.6}$$

Also, it is easily observed that

$$\lim_{s\to\frac{z+2}{2}}\left(s-\frac{z+2}{2}\right)\zeta\left(s-\frac{z}{2}\right)=1. \tag{3.7}$$

Thus using 3.6) and (3.7), we see that

$$\begin{aligned}R_{\frac{z+2}{2}}&=\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{z+1}{2}\right)\zeta(z+1)\left(\frac{\pi}{y}\right)^{\left(-\frac{z+2}{2}\right)}\\&=y^{1+\frac{z}{2}}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z).\end{aligned} \tag{3.8}$$

Also,

$$\lim_{s\to\frac{2-z}{2}}\left(s-\frac{2-z}{2}\right)\zeta\left(s+\frac{z}{2}\right)=1. \tag{3.9}$$

Hence using (3.6) and (3.9), we deduce that

$$\begin{aligned}R_{\frac{2-z}{2}}&=\Gamma\left(\frac{1-z}{2}\right)\Gamma\left(\frac{1}{2}\right)\zeta(1-z)\left(\frac{\pi}{y}\right)^{\left(-\frac{2-z}{2}\right)}\\&=y^{1-\frac{z}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z).\end{aligned} \tag{3.10}$$

Next, we show that as $T\to\infty$, the integrals along the horizontal segments $[\frac{1}{2}-iT,\frac{3}{2}-iT]$ and $[\frac{3}{2}+iT,\frac{1}{2}+iT]$ tend to zero. We use the following estimate for $\zeta(s)$ valid for $\sigma\geq-\delta$ [12, p. 95, eqn. 5.1.1],

$$\zeta(s)=O(t^{\frac{3}{2}+\delta}). \tag{3.11}$$

Since $\frac{1}{2}<\text{Re } s<\frac{3}{2}$ and $-1<\text{Re } z<1$, we readily see that $0<\text{Re }\left(s-\frac{z}{2}\right)<2$ and $0<\text{Re }\left(s+\frac{z}{2}\right)<2$. So if $s=\sigma+it$, $z=x+iy$ and δ is any positive number, then using (3.2) and (3.11), we find that

$$\begin{aligned}&\left|\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds\right|\\&\leq C(2\pi)\left|\frac{T}{2}-\frac{y}{4}\right|^{\frac{\sigma-1}{2}-\frac{x}{4}}\left|\frac{T}{2}+\frac{y}{4}\right|^{\frac{\sigma-1}{2}+\frac{x}{4}}e^{-\frac{\pi}{2}\left(\left|\frac{T}{2}-\frac{y}{4}\right|+\left|\frac{T}{2}+\frac{y}{4}\right|\right)}\left(T-\frac{y}{2}\right)^{\frac{3}{2}+\delta}\left(T+\frac{y}{2}\right)^{\frac{3}{2}+\delta}\\&\quad\times\left(1+O\left(\frac{1}{\left|\frac{T}{2}-\frac{y}{4}\right|}\right)\right)\left(1+O\left(\frac{1}{\left|\frac{T}{2}+\frac{y}{4}\right|}\right)\right)\\&=o(1),\end{aligned} \tag{3.12}$$

as T tends to infinity, where C is some constant. Thus,

$$\lim_{T\to\infty}\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds=0. \tag{3.13}$$

Similarly we can show that

$$\lim_{T\to\infty}\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds=0. \tag{3.14}$$

Now it remains to evaluate

$$\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s} ds. \tag{3.15}$$

From [6, p. 115, formula 11.1], for $c = \text{Re } s > \pm \text{ Re } \nu$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s-2}a^{-s}\Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)x^{-s}\, ds = K_{\nu}(ax). \tag{3.16}$$

Interchanging the order of summation and integration because of absolute convergence in (3.15), using (3.4) and then using (3.16) with $\nu = \frac{z}{2}$, $a = 2$ and $x = \pi m/y$, we see that since Re $s = \frac{3}{2} >$ Re $\frac{z}{2}$, we have

$$\begin{aligned}
&\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s} ds\\
&=\sum_{m=1}^{\infty}\sigma_{-z}(m)m^{\frac{z}{2}}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\left(\frac{\pi m}{y}\right)^{-s} ds\\
&=8\pi i\sum_{m=1}^{\infty}\sigma_{-z}(m)m^{\frac{z}{2}}K_{\frac{z}{2}}\left(\frac{2\pi m}{y}\right).
\end{aligned} \tag{3.17}$$

Thus from (3.3), (3.5), (3.8), (3.10), (3.13), (3.14) and (3.17), we see that with $y = e^{2n}$,

$$\begin{aligned}
&64\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos nt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt\\
&=\frac{1}{i\sqrt{y}}\left(8\pi i\sum_{m=1}^{\infty}\sigma_{-z}(m)m^{\frac{z}{2}}K_{\frac{z}{2}}\left(\frac{2\pi m}{y}\right)-2\pi i y^{1-\frac{z}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)-2\pi i y^{1+\frac{z}{2}}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\right)\\
&=-\frac{2\pi}{\sqrt{y}}\left(y^{1-\frac{z}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+y^{1+\frac{z}{2}}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)-4\sum_{m=1}^{\infty}\sigma_{-z}(m)m^{\frac{z}{2}}K_{\frac{z}{2}}\left(\frac{2\pi m}{y}\right)\right).
\end{aligned} \tag{3.18}$$

Now let $n = \frac{1}{2}\log\alpha$. Then $y = e^{2n}$ implies that $y = \alpha$. Since $\alpha\beta = 1$, (3.18) becomes

$$\begin{aligned}
&-\frac{32}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt\\
&=\sqrt{\beta}\left(\beta^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\beta^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)-4\sum_{m=1}^{\infty}\sigma_{-z}(m)m^{z/2}K_{\frac{z}{2}}\left(2m\pi\beta\right)\right).
\end{aligned} \tag{3.19}$$

Finally, switching the roles of α and β in (3.19) and then combining the result with (3.19), we arrive at (1.13), since with $\alpha\beta = 1$, the left-hand side of (3.19) is invariant under the map $\alpha \to \beta$. This completes the proof.

Remarks. 1. It can be readily seen from (1.13) that the identity is invariant if we replace z by $-z$.

2. The first equality in (1.13) can be easily simplified to obtain (1.11), where $\alpha\beta = \pi^2$.

As a corollary of Theorem 1.4, we get an extension of Koshliakov's formula given in [5].

Corollary 3.1 (Extended version of Koshliakov's formula)**.** *Let $d(n)$ denote the number of positive divisors of n and let $K_s(n)$ be defined as before. If α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\sqrt{\alpha}\left(\frac{\gamma-\log(4\pi\alpha)}{\alpha}-4\sum_{n=1}^{\infty}d(n)K_0(2\pi n\alpha)\right)=\sqrt{\beta}\left(\frac{\gamma-\log(4\pi\beta)}{\beta}-4\sum_{n=1}^{\infty}d(n)K_0(2\pi n\beta)\right)$$
$$=-\frac{32}{\pi}\int_0^{\infty}\frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2\cos\left(\frac{1}{2}t\log\alpha\right)\,dt}{(1+t^2)^2}. \tag{3.20}$$

Proof. Let $z \to 0$ in Theorem 1.4. Using Lebesgue's dominated convergence theorem, we can interchange the limit and the integral sign readily giving the integral in (3.20). For obtaining the first equality in (3.20), we follow along the similar lines as in the proof of Corollary 3.4 in [2], i.e., by using the following series expansions [7, p. 944, formula 8.321, no. 1]

$$\Gamma(z)=\frac{1}{z}-\gamma+\cdots, \tag{3.21}$$

and [12, pp. 19–20, eqns. (2.4.3), (2.4.5)]

$$\zeta(z)=-\frac{1}{2}-\frac{1}{2}\log(2\pi)z+\cdots, \tag{3.22}$$

along with

$$\alpha^{z/2}=e^{\frac{z}{2}\log\alpha}=1+\frac{z}{2}\log\alpha+\frac{z^2}{4.2!}(\log\alpha)^2+\cdots, \tag{3.23}$$

and also noting that $\lim_{z\to 0}\sigma_{-z}(n)=d(n)$. □

Using arguments similar to those in the proof of Theorem 1.4, we obtain the following analogue of Theorem 1.4. For a detailed proof, see the author's doctoral thesis [4].

Theorem 3.2. *If α and β are positive numbers such that $\alpha\beta = 1$, then for Re $z > 1$,*

$$\sqrt{\alpha}\left(-\alpha^{-\frac{z}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\alpha^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\alpha\right)\right)$$
$$=\sqrt{\beta}\left(-\beta^{-\frac{z}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\beta^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\beta\right)\right)$$
$$=-\frac{32}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt. \tag{3.24}$$

Observe that if we replace z by $-z$, we obtain the following corollary.

Corollary 3.3. *If α and β are positive numbers such that $\alpha\beta = 1$, then for Re $z < -1$,*

$$\sqrt{\alpha}\left(\alpha^{\frac{z}{2}-1}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) - \alpha^{\frac{z}{2}}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z) - 4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\alpha\right)\right)$$
$$= \sqrt{\beta}\left(\beta^{\frac{z}{2}-1}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) - \beta^{\frac{z}{2}}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z) - 4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\beta\right)\right)$$
$$= -\frac{32}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt. \tag{3.25}$$

4. Modular relations involving the Hurwitz zeta function

In this section, we first give a proof of Theorem 1.5 and then the proofs of Theorem 1.6 and Theorem 1.7 respectively.

Proof of Theorem 1.5. Let

$$f(z,t) = \frac{1}{(4t^2+(z+1)^2)}\Gamma\left(\frac{z-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{z-1}{4}-\frac{it}{2}\right), \tag{4.1}$$

so that $\phi(z,s) = \dfrac{1}{(2s+z+1)}\Gamma\left(\dfrac{z-1}{4}+\dfrac{s}{2}\right)$. Then by an application of (2.8) with the above f and ϕ, we observe that

$$\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt$$
$$= \frac{1}{4i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{1}{\left(\frac{z}{2}+s\right)}\Gamma\left(\frac{z-2}{4}+\frac{s}{2}\right)\frac{1}{\left(\frac{z+2}{2}-s\right)}\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)y^s\,ds$$
$$= -\frac{1}{16i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)$$
$$\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds, \tag{4.2}$$

using (1.1) in the penultimate line.

To evaluate the integral in the last step in (4.2), we want to use the series representation for $\zeta\left(s+\frac{z}{2}\right)$, namely,

$$\zeta\left(s+\frac{z}{2}\right) = \sum_{m=1}^{\infty}\frac{1}{m^{s+\frac{z}{2}}}, \tag{4.3}$$

valid for Re $\left(s+\frac{z}{2}\right) > 1$, i.e., Re $s >$ Re $\left(\frac{2-z}{2}\right)$. But since $-1 <$ Re $z < 1$, we have $0 <$ Re $\left(s+\frac{z}{2}\right) < 1$. Thus we move the line of integration from Re $s = \frac{1}{2}$ to Re $s = \frac{3}{2}$. In this process, we encounter a pole of order 1 at $s = \frac{z+2}{2}$ (due to $\zeta(s-\frac{z}{2})$) and a pole of order 1 at $s = \frac{2-z}{2}$ (due to $\Gamma(\frac{s}{2}+\frac{z-2}{4})$). Note that the pole of $\zeta\left(s+\frac{z}{2}\right)$ at $s = \frac{2-z}{2}$ is cancelled by the zero of $\left(s+\frac{z-2}{2}\right)$ at $s=\frac{2-z}{2}$, since

$$\lim_{s\to\frac{2-z}{2}}\left(s+\frac{z-2}{2}\right)\zeta\left(s+\frac{z}{2}\right)=1. \tag{4.4}$$

Let $T>0$ denote a real number. Then by residue theorem and using the notation in (1.21), we know that

$$\begin{aligned}
&\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\\
&\qquad\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds\\
&=\left[\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}\right]\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\\
&\qquad\times\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds\\
&\quad-2\pi i\left(R_{\frac{z+2}{2}}+R_{\frac{2-z}{2}}\right).
\end{aligned} \tag{4.5}$$

We evaluate the first residue above by using the functional equation of the Gamma function

$$\Gamma(z+1)=z\Gamma(z), \tag{4.6}$$

Legendre's duplication formula

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2s-1}}\Gamma(2s), \tag{4.7}$$

and the facts that $\Gamma(\frac{1}{2})=\sqrt{\pi}$ and

$$\lim_{s\to\frac{z+2}{2}}\left(s-\frac{z+2}{2}\right)\zeta\left(s-\frac{z}{2}\right)=1. \tag{4.8}$$

Thus,

$$\begin{aligned}
R_{\frac{z+2}{2}}&=4\lim_{s\to\frac{z+2}{2}}\left(s-\frac{z+2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}+1\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\\
&\qquad\times\Gamma\left(\frac{s}{2}-\frac{z}{4}+1\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}\\
&=4\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{z}{2}+1\right)\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{z+1}{2}\right)\zeta(z+1)\left(\frac{\pi}{y}\right)^{-\frac{z}{2}-1}\\
&=-2^{2-z}\pi^{\frac{1-z}{2}}y^{1+\frac{z}{2}}\Gamma(z+1)\zeta(z+1).
\end{aligned} \tag{4.9}$$

For the second limit, we use the functional equation for the Riemann zeta function in the asymmetric form [12, p. 13, eqn. (2.1.1)],

$$\zeta(s)=2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\tfrac{1}{2}\pi s\right), \tag{4.10}$$

and the reflection formula for the Gamma function

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}. \tag{4.11}$$

Then we have

$$\begin{aligned}
R_{\frac{2-z}{2}} &= 4\lim_{s\to\frac{2-z}{2}}\left(s-\frac{2-z}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}+1\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\\
&\qquad\times\Gamma\left(\frac{s}{2}-\frac{z}{4}+1\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}\\
&= 4\Gamma\left(\frac{3-z}{2}\right)\Gamma\left(\frac{z-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\zeta(1-z)\pi^{\frac{z}{2}-1}\left(\frac{\pi}{y}\right)^{\frac{z}{2}-1}\\
&\quad -4\frac{\pi}{\cos\left(\frac{1}{2}\pi z\right)}\sqrt{\pi}2^{1-z}\pi^{-z}\Gamma(z)\zeta(z)\sin\left(\frac{1}{2}\pi(1-z)\right)\pi^{\frac{z}{2}-1}y^{1-\frac{z}{2}}\\
&= -2^{3-z}\pi^{\frac{1-z}{2}}y^{1-\frac{z}{2}}\Gamma(z)\zeta(z).
\end{aligned} \tag{4.12}$$

It can be easily seen, by the use of Stirling's formula (3.2), that as $T\to\infty$, the integrals along the horizontal segments $[\frac{1}{2}-iT,\frac{3}{2}-iT]$ and $[\frac{3}{2}+iT,\frac{1}{2}+iT]$ tend to zero, i.e.,

$$\begin{aligned}
\lim_{T\to\infty}\int_{\frac{1}{2}\pm iT}^{\frac{3}{2}\pm iT}&\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\\
&\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds=0.
\end{aligned} \tag{4.13}$$

It remains to evaluate

$$\begin{aligned}
\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}&\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\\
&\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds.
\end{aligned} \tag{4.14}$$

We first simplify the integrand using first (4.7) and (4.11). Thus,

$$\begin{aligned}
&\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\\
&\qquad\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds\\
&=4\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma\left(\frac{s}{2}+\frac{z}{4}+\frac{1}{2}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(1+\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\\
&\qquad\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds\\
&=\frac{4\pi^{\frac{3}{2}}}{2^{\frac{z}{2}-1}}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{2}}}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\frac{\Gamma\left(s+\frac{z}{2}\right)\zeta\left(s-\frac{z}{2}\right)}{2^{s}\sin\left(\pi\left(\frac{z}{4}-\frac{s}{2}\right)\right)}\left(\frac{\pi m}{y}\right)^{-s}ds, \qquad (4.15)
\end{aligned}$$

where we have interchanged the order of summation and integration because of absolute convergence. Using (4.10) for $\zeta(1-z)$, we find that

$$\begin{aligned}
&\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\\
&\qquad\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds\\
&=\frac{4\pi^{\frac{3}{2}}}{2^{\frac{z}{2}-1}}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{2}}}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\frac{2^{s-\frac{z}{2}}\pi^{s-\frac{z}{2}-1}\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\sin\left(\frac{\pi}{2}\left(s-\frac{z}{2}\right)\right)\Gamma\left(s+\frac{z}{2}\right)\zeta\left(s-\frac{z}{2}\right)}{2^{s}\sin\left(\pi\left(\frac{z}{4}-\frac{s}{2}\right)\right)}\\
&\qquad\times\left(\frac{\pi m}{y}\right)^{-s}ds\\
&=-2^{3-z}\pi^{\frac{1-z}{2}}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{2}}}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s}ds. \qquad (4.16)
\end{aligned}$$

Now from [6, p. 203, formula 5.84], we see that for $0<c=\text{Re } s<\text{Re } \nu-1$ and Re $\alpha>0$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}a^{-s}\Gamma(s)\Gamma(\nu-s)\zeta(\nu-s,\alpha)x^{-s}\,ds=\Gamma(\nu)\zeta(\nu,\alpha+ax). \qquad (4.17)$$

Let $\nu=z+1$, $a=1$, $\alpha=1$ and $x=m/y$. Then noting that $\zeta(s,1)=\zeta(s)$, for $0<c=$ Re $s<$ Re z,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(z+1-s)\zeta(z+1-s)\left(\frac{m}{y}\right)^{-s}ds=\Gamma(z+1)\zeta\left(z+1,1+\frac{m}{y}\right). \qquad (4.18)$$

Next, replacing s by $s+\frac{z}{2}$, we see that for $-\operatorname{Re}\left(\frac{z}{2}\right)<\operatorname{Re} s<\operatorname{Re}\left(\frac{z}{2}\right)$,

$$\begin{aligned}&\int_{c-\operatorname{Re}\left(\frac{z}{2}\right)-i\infty}^{c-\operatorname{Re}\left(\frac{z}{2}\right)+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s}ds\\&=2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\zeta\left(z+1,1+\frac{m}{y}\right).\end{aligned}\tag{4.19}$$

Hence in order to use (4.19) to simplify the integral in the last expression in (4.16), we shift the line of integration from $\operatorname{Re} s=c-\operatorname{Re}\left(\frac{z}{2}\right)$ to $\operatorname{Re} s=\frac{3}{2}$. In doing that, we encounter a pole of order 1 at $s=\frac{z}{2}$ (due to $\zeta\left(1-s+\frac{z}{2}\right)$) and a pole of order 1 at $s=1+\frac{z}{2}$ (due to $\Gamma\left(1-s+\frac{z}{2}\right)$). Thus by another application of the residue theorem, we see that

$$\begin{aligned}&\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s}ds\\&=\int_{c-\operatorname{Re}\left(\frac{z}{2}\right)-i\infty}^{c-\operatorname{Re}\left(\frac{z}{2}\right)+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s}ds\\&\quad+2\pi i\Bigg(\lim_{s\to\frac{z}{2}}\left(s-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s}\\&\quad+\lim_{s\to1+\frac{z}{2}}\left(s-1-\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s}\Bigg)\\&=\int_{c-\operatorname{Re}\left(\frac{z}{2}\right)-i\infty}^{c-\operatorname{Re}\left(\frac{z}{2}\right)+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s}ds\\&\quad+2\pi i\left(-\left(\frac{m}{y}\right)^{-\frac{z}{2}}\Gamma(z)+\frac{1}{2}\left(\frac{m}{y}\right)^{-1-\frac{z}{2}}\Gamma(1+z)\right).\end{aligned}\tag{4.20}$$

Now we make use of the simple fact that

$$\zeta(s,a+1)=\zeta(s,a)-a^{-s},\tag{4.21}$$

which is easily seen first for $\operatorname{Re} s>1$, and then for all s by analytic continuation. Then from (4.19), (4.20) and (4.21), we deduce that

$$\begin{aligned}&\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s}ds\\&=2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\left(\zeta\left(z+1,\frac{m}{y}\right)-\frac{1}{(m/y)^{z+1}}+\frac{1}{2\,(m/y)^{z+1}}-\frac{(m/y)^{-z}}{z}\right)\\&=2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\left(\zeta\left(z+1,\frac{m}{y}\right)-\frac{1}{2\,(m/y)^{z+1}}+\frac{(m/y)^{-z}}{-z}\right)\\&=2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\varphi\left(z,\frac{m}{y}\right),\end{aligned}\tag{4.22}$$

where $\varphi(z,x)$ is defined in (1.15). Thus from (4.16) and (4.22), we see that

$$\begin{aligned}&\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\\&\qquad\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds\\&=-2\pi i2^{3-z}\pi^{\frac{1-z}{2}}y^{-\frac{z}{2}}\Gamma(z+1)\sum_{m=1}^{\infty}\varphi\left(z,\frac{m}{y}\right).\end{aligned}\tag{4.23}$$

From (4.2), (4.5), (4.9), (4.12), (4.13) and (4.23), we observe that

$$\begin{aligned}&\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt\\&=-\frac{1}{16i\sqrt{y}}\Bigg(-2\pi i2^{3-z}\pi^{\frac{1-z}{2}}y^{-\frac{z}{2}}\Gamma(z+1)\sum_{m=1}^{\infty}\varphi\left(z,\frac{m}{y}\right)-2\pi i\Big(-2^{2-z}\pi^{\frac{1-z}{2}}y^{1+\frac{z}{2}}\Gamma(z+1)\zeta(z+1)\\&\qquad-2^{3-z}\pi^{\frac{1-z}{2}}y^{1-\frac{z}{2}}\Gamma(z)\zeta(z)\Big)\Bigg)\\&=2^{-z}\pi^{\frac{3-z}{2}}y^{-\frac{(z+1)}{2}}\Gamma(z+1)\left(\sum_{m=1}^{\infty}\varphi\left(z,\frac{m}{y}\right)-\frac{y^{z+1}\zeta(z+1)}{2}-\frac{y\zeta(z)}{z}\right).\end{aligned}\tag{4.24}$$

Now let $n=\frac{1}{2}\log\alpha$ in (4.24) so that $y=\alpha$. Since $\alpha\beta=1$, (4.24) becomes

$$\begin{aligned}&\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt\\&=\beta^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\varphi(z,n\beta)-\frac{\zeta(z+1)}{2\beta^{z+1}}-\frac{\zeta(z)}{\beta z}\right).\end{aligned}\tag{4.25}$$

Finally, switching the roles of α by β in (4.25) and then combining the result with (4.25), we arrive at (1.16), since with $\alpha\beta=1$, the left-hand side of (4.25) is invariant under the map $\alpha\to\beta$. This completes the proof. □

Remark. Ramanujan's transformation formula (1.6) can be easily obtained by letting $z\to 0$ in (1.16) and then using Lebesgue's dominated convergence theorem. For details, see [3].

Proof of Theorem 1.6. Since the right-hand side of (1.17) is exactly the same as that in (1.16), the proof as well as many of the calculations in Theorem 1.6 are similar, in fact simpler, than that of Theorem 1.5, and so we will be brief.

We use (4.2) again. Since Re $z>1$, we see that Re $\left(s+\frac{z}{2}\right)>1$ and thus we can directly use (4.3). Hence,

$$\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt$$

$$\times \zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s} ds$$
$$= -\frac{1}{16i\sqrt{y}}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{2}}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)$$
$$\times \Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\left(\frac{\pi m}{y}\right)^{-s} ds, \tag{4.26}$$

where we have interchanged the order of summation and integration because of absolute convergence. Then, using the exact same method as in (4.15)-(4.16) to simplify the integrand in the last step in (4.26), we see that

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)$$
$$\times \Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\left(\frac{\pi m}{y}\right)^{-s} ds$$
$$= -2^{3-z}\pi^{\frac{1-z}{2}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds. \tag{4.27}$$

Next, we want to use (4.19) to simplify the inner integral on the right-hand side of (4.26). Thus we need to shift the line of integration from Re $s = c-\text{Re}\left(\frac{z}{2}\right)$ to the line Re $s = \frac{1}{2}$. But unlike in the previous case $-1 < \text{Re } z < 1$, here we do not encounter any poles of the integrand on the left-hand side of (4.19) in the shifting process. Thus for $c = \text{Re } s$,

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds$$
$$= \int_{c-\text{Re}\left(\frac{z}{2}\right)-i\infty}^{c-\text{Re}\left(\frac{z}{2}\right)+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds$$
$$= 2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\zeta\left(z+1, 1+\frac{m}{y}\right). \tag{4.28}$$

Then from (4.26), (4.27) and (4.28), we find that

$$\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2} dt$$
$$= -\frac{1}{16i\sqrt{y}}\left(-2^{3-z}\pi^{\frac{1-z}{2}}\sum_{m=1}^{\infty}\frac{2\pi i}{m^{\frac{z}{2}}}\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\zeta\left(z+1, 1+\frac{m}{y}\right)\right)$$
$$= \frac{1}{8}(4\pi)^{-\frac{(z-3)}{2}}y^{-\frac{(z+1)}{2}}\Gamma(z+1)\sum_{m=1}^{\infty}\zeta\left(z+1, 1+\frac{m}{y}\right). \tag{4.29}$$

Now letting $n=\frac{1}{2}\log\alpha$ in (4.29) so that $y=\alpha$, we observe that

$$\frac{8(4\pi)^{\frac{(z-3)}{2}}}{\Gamma(z+1)}\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt$$
$$=\alpha^{-\frac{(z+1)}{2}}\sum_{m=1}^\infty \zeta\left(z+1,1+\frac{m}{\alpha}\right). \qquad (4.30)$$

Now replacing α by β in (4.30) and then combining the result with (4.30), we arrive at (1.17) since the left-hand side of (4.30) is invariant under the map $\alpha\to\beta$. This completes the proof. □

Proof of Theorem 1.7. The proof of (1.19) is along the lines similar to those in the proof of (1.16), and so we will be brief.

We want to evaluate the integral in the last step in (4.2) by making use of (4.3), valid for Re $\left(s+\frac{z}{2}\right)>1$, i.e., Re $s>$ Re $\left(\frac{2-z}{2}\right)$. But since $-3<$ Re $z<-1$, we have $-1<$ Re $\left(s+\frac{z}{2}\right)<0$. While shifting the line of integration from Re $s=\frac{1}{2}$ to Re $s=\frac{5}{2}$, we encounter poles of order 1 at $s=\frac{2-z}{2}$, $s=\frac{z+4}{2}$ and $s=-\frac{z}{2}$. Let $T>0$ denote a real number. We apply residue theorem by considering the rectangle with segments $[\frac{1}{2}-iT,\frac{5}{2}-iT]$, $[\frac{5}{2}-iT,\frac{5}{2}+iT]$, $[\frac{5}{2}+iT,\frac{1}{2}+iT]$ and $[\frac{1}{2}+iT,\frac{1}{2}-iT]$. The residues at the above poles can be calculated similarly as before and so the calculations are suppressed here. Thus using the notation in (1.21), we find that

$$R_{\frac{2-z}{2}}=-2^{3-z}\pi^{\frac{1-z}{2}}y^{1-\frac{z}{2}}\Gamma(z)\zeta(z), \qquad (4.31)$$

$$R_{\frac{z+4}{2}}=\frac{1}{3}2^{1-z}\pi^{\frac{1-z}{2}}y^{2+\frac{z}{2}}\Gamma(z+2)\zeta(z+2), \qquad (4.32)$$

and

$$R_{\frac{-z}{2}}=-4\pi^{\frac{3+z}{2}}y^{-\frac{z}{2}}\frac{\zeta(z)}{\sin\left(\frac{1}{2}\pi z\right)}. \qquad (4.33)$$

As $T\to\infty$, the integrals along the horizontal segments $[\frac{1}{2}-iT,\frac{5}{2}-iT]$ and $[\frac{5}{2}+iT,\frac{1}{2}+iT]$ tend to zero. Next, doing the exact same calculations as in (4.15) and (4.16), we have

$$\int_{\frac{5}{2}-i\infty}^{\frac{5}{2}+i\infty}\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)$$
$$\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}ds$$
$$=-2^{3-z}\pi^{\frac{1-z}{2}}\sum_{m=1}^\infty\frac{1}{m^{\frac{z}{2}}}\int_{\frac{5}{2}-i\infty}^{\frac{5}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s}ds. \qquad (4.34)$$

In order to use (4.19) to simplify the integral in the last expression in (4.34), we shift the line of integration from Re $s=c-$ Re $\left(\frac{z}{2}\right)$ to Re $s=\frac{5}{2}$ and then from another application of the residue theorem, we see after simplification that

$$\int_{\frac{5}{2}-i\infty}^{\frac{5}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s}ds$$
$$=2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\zeta\left(z+1,1+\frac{m}{y}\right)$$
$$+2\pi i\left(-\left(\frac{m}{y}\right)^{-\frac{z}{2}}\Gamma(z)+\frac{1}{2}\left(\frac{m}{y}\right)^{-1-\frac{z}{2}}\Gamma(1+z)-\frac{1}{12}\left(\frac{m}{y}\right)^{-2-\frac{z}{2}}\Gamma(2+z)+0\right)$$
$$=2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\left(\zeta\left(z+1,\frac{m}{y}\right)-\frac{(m/y)^{-1-\frac{z}{2}}}{2}+\frac{(m/y)^{-\frac{z}{2}}}{-z}-\frac{(z+1)(m/y)^{-2-\frac{z}{2}}}{12}\right)$$
$$=2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\Lambda\left(z,\frac{m}{y}\right), \tag{4.35}$$

where we used (4.21) in the penultimate step and (1.18) in the last step. Finally from (4.2), (4.31), (4.32), (4.33), (4.34) and (4.35), we find after simplification that

$$\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^\infty\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt$$
$$=y^{-\frac{(z+1)}{2}}\left(\sum_{m=1}^\infty\Lambda\left(z,\frac{m}{y}\right)-\frac{y\zeta(z)}{z}+\frac{\zeta(z+1)}{2}+\frac{(z+1)y^{z+2}\zeta(z+2)}{12}\right). \tag{4.36}$$

Now letting $n=\frac{1}{2}\log\alpha$ in (4.36) so that $y=\alpha$, we observe that since $\alpha\beta=1$,

$$\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^\infty\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt$$
$$=\beta^{\frac{z+1}{2}}\left(\sum_{n=1}^\infty\Lambda(z,n\beta)-\frac{\zeta(z)}{\beta z}+\frac{\zeta(z+1)}{2}+\frac{(z+1)\zeta(z+2)}{12\beta^{z+2}}\right). \tag{4.37}$$

Finally, switching the roles of α by β in (4.37) and then combining the result with (4.37), we arrive at (1.19), since with $\alpha\beta=1$, the left-hand side of (4.37) is invariant under the map $\alpha\to\beta$. This completes the proof.

□

Acknowledgements. The author wishes to thank Professor Bruce C. Berndt for his constant support, careful reading of this manuscript and for helpful comments.

References

[1] B.C. Berndt and A. Dixit, *A transformation formula involving the Gamma and Riemann zeta functions in Ramanujan's Lost Notebook*, to appear in *The legacy of Alladi Ramakrishnan in the mathematical sciences*, K. Alladi, J. Klauder, C. R. Rao, Eds, Springer, New York, 2010.

[2] B.C. Berndt, Y. Lee, and J. Sohn, *The formulas of Koshliakov and Guinand in Ramanujan's lost notebook*, Surveys in Number Theory, Series: Developments in Mathematics, vol. 17, K. Alladi, ed., Springer, New York, 2008, pp. 21–42.

[3] A. Dixit, *Analogues of a transformation formula of Ramanujan*, submitted for publication.

[4] A. Dixit, *Doctoral Thesis*, University of Illinois at Urbana-Champaign.

[5] A. Dixit, *Series transformations and integrals involving the Riemann Ξ-function*, submitted for publication.
[6] F. Oberhettinger, *Tables of Mellin Transforms*, Springer-Verlag, New York, 1974.
[7] I. S. Gradshteyn and I. M. Ryzhik, eds., *Table of Integrals, Series, and Products*, 5th ed., Academic Press, San Diego, 1994.
[8] A. P. Guinand, *Some rapidly convergent series for the Riemann ξ-function*, Quart. J. Math. (Oxford) **6** (1955), 156–160.
[9] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer, Berlin, 1966.
[10] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
[11] S. Ramanujan, *New expressions for Riemann's functions $\xi(s)$ and $\Xi(t)$*, Quart. J. Math. **46** (1915), 253–260.
[12] E.C. Titchmarsh, *The Theory of the Riemann Zeta Function*, Clarendon Press, Oxford, 1986.
[13] G. N. Watson, *Some self-reciprocal functions*, Quart. J. Math. (Oxford) **2** (1931), 298–309.

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: aadixit2@illinois.edu