THE UNIMODALITY OF A POLYNOMIAL COMING FROM A RATIONAL INTEGRAL. BACK TO THE ORIGINAL PROOF

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ABSTRACT. A sequence of coefficients that appeared in the evaluation of a rational integral has been shown to be unimodal. An alternative proof is presented.

1. INTRODUCTION

The polynomial

$$P_m(a) = \sum_{\ell=0}^{m} d_\ell(m) a^\ell \tag{1.1}$$

with

$$d_\ell(m) = 2^{-2m} \sum_{k=\ell}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{\ell} \tag{1.2}$$

made its appearance in [1] in the evaluation of the quartic integral

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a). \tag{1.3}$$

Properties of the sequence of numbers $\{d_\ell(m)\}$ are discussed in [9]. Among them is the fact that this is a unimodal sequence. Recall that a sequence of real numbers $\{x_0, x_1, \cdots, x_m\}$ is called *unimodal* if there exists an index $0 \leq j \leq m$ such that $x_0 \leq x_1 \leq \cdots \leq x_j$ and $x_j \geq x_{j+1} \geq \cdots x_m$. The sequence is called *logconcave* if $x_j^2 \geq x_{j-1}x_{j+1}$ for $1 \leq j \leq m-1$. It is easy to see that if a sequence is logconcave then it is unimodal [13].

The sequence $\{d_\ell(m)\}$ was shown to be unimodal in [2] by an elementary argument and it was conjectured there to be logconcave. This conjecture was established by M. Kauers and P. Paule [8] using four recurrence relations found using a computer algebra approach. W. Y. Chen and E. X. W. Xia [6] introduced the notion of *ratio-monotonicity* for a sequence $\{x_m\}$:

$$\frac{x_0}{x_{m-1}} \leq \frac{x_1}{x_{m-2}} \leq \cdots \leq \frac{x_i}{x_{m-1-i}} \leq \cdots \leq \frac{x_{\lfloor \frac{m}{2} \rfloor - 1}}{x_{m-\lfloor \frac{m}{2} \rfloor}} \leq 1. \tag{1.4}$$

The results in [6] show that $\{d_\ell(m)\}$ is a ratio-monotone sequence and, as can be easily checked, this implies the logconcavity of $\{d_\ell(m)\}$. The logconcavity of

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$\{d_\ell(m)\}$ also follows from the *minimum conjecture* stated in [10]: let $b_\ell(m) = 2^{2m} d_\ell(m)$. The function

$$(m+\ell)(m+1-\ell)b_{\ell-1}^2(m) + \ell(\ell+1)b_\ell^2(m) - \ell(2m+1)b_{\ell-1}(m),$$

defined for $1 \leq \ell \leq m$, attains its minimum at $\ell = m$ with value $2^{2m} m(m+1)\binom{2m}{m}^2$. This has been proven in [7], providing an alternative proof of the logconcavity of $\{d_\ell(m)\}$.

Further study of the sequence $\{d_\ell(m)\}$ are defined in terms of the operator

$$\mathfrak{L}\left(\{x_k\}\right) = \left\{x_k^2 - x_{k-1}x_{k+1}\right\}. \tag{1.5}$$

For instance, $\{x_k\}$ is logconcave simply means $\mathfrak{L}\left(\{x_k\}\right)$ is a nonnegative sequence. The sequence is called i-logconcave if $\mathfrak{L}^j\left(\{x_k\}\right)$ is a nonnegative sequence for $0 \leq j \leq i$. A sequence that is i-logconcave for every $i \in \mathbb{N}$ is called *infinitely logconcave.*

Conjecture 1.1. *The sequence $\{d_\ell(m)\}$ is infinitely logconcave.*

There is a strong connection between the roots of a polynomial $P(x)$ and ordering properties of its coefficients. For instance, if $P(x)$ has only real negative zeros, then P is logconcave (see [13] for details). Therefore, the expansion of $(x+1)^n$ shows that the binomial coefficients form a logconcave sequence. P. Brändén [3] showed that if $P(x) = a_0 + a_1x + \cdots + a_nx^n$, with $a_j \geq 0$ has only real roots, then the same is true for

$$P_1(x) = a_0^2 + (a_1^2 - a_0a_2)x + \cdots + (a_{n-1}^2 - a_{n-2}a_n)x^n. \tag{1.6}$$

This implies that the binomial coefficients are infinitely logconcave. This approach fails with the sequence $\{d_\ell(m)\}$ since the polynomial $P_m(a)$ has mostly non-real zeros. On the other hand, Brändén conjectured and W. Y. C. Chen et al [5] proved that $Q_m(x) = \sum_{\ell=0}^{m} \frac{d_\ell(m)}{\ell!}x^\ell$ and $R_m(x) = \sum_{\ell=0}^{m} \frac{d_\ell(m)}{(\ell+2)!}x^\ell$ have only real zeros. These results imply that $P_m(a)$ in (1.1) is 3-logconcave.

The goal of this paper is to present an improved version of the original proof of the theorem

Theorem 1.2. *The sequence $\{d_\ell(m)\}$ is unimodal.*

The proof of Theorem 1.2 given in [2] is based on the difference

$$\Delta d_\ell(m) = d_{\ell+1}(m) - d_\ell(m). \tag{1.7}$$

A simple calculation shows that

$$\Delta d_\ell(m) = \frac{1}{2^{2m}}\binom{m+\ell}{m}\sum_{k=\ell}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m+\ell} \times \frac{k-2\ell-1}{\ell+1}. \tag{1.8}$$

For $\left\lfloor \frac{m}{2} \right\rfloor \leq \ell \leq m-1$, the inequality

$$k - 2\ell - 1 \leq k - 2\left\lfloor \frac{m}{2} \right\rfloor - 1 \leq k - m \leq 0 \tag{1.9}$$

shows that $\Delta d_\ell(m) < 0$ since the term for $k = \ell$ has a strictly negative contribution. In the range $0 \leq \ell < \left\lfloor \frac{m}{2} \right\rfloor$, the difference $\Delta d_\ell(m) > 0$. This is equivalent to

(1.10)

$$\sum_{k=\ell}^{2\ell} 2^k(2\ell+1-k)\binom{2m-2k}{m-k}\binom{m+k}{m+\ell} < \sum_{k=2\ell+2}^{m} 2^k(k-2\ell-1)\binom{2m-2k}{m-k}\binom{m+k}{m+\ell}.$$

Fact 1. The inequality (1.10) implies Theorem 1.2.

This required inequality is valid in an even stronger form, obtained by replacing $k-2\ell-1$ on the right hand side of (1.10) by 1 to produce

$$\sum_{k=\ell}^{2\ell} 2^k(2\ell+1-k)\binom{2m-2k}{m-k}\binom{m+k}{m+\ell} < \sum_{k=2\ell+2}^{m} 2^k\binom{2m-2k}{m-k}\binom{m+k}{m+\ell}, \tag{1.11}$$

and then made even stronger by replacing the sum on the right hand side of (1.11) by its last term. Therefore, if

$$\sum_{k=\ell}^{2\ell} 2^k(2\ell+1-k)\binom{2m-2k}{m-k}\binom{m+k}{m+\ell} < 2^m\binom{2m}{m+\ell}, \tag{1.12}$$

then $\Delta d_\ell(m) > 0$. This last inequality is now written as

$$S_{m,\ell} := \sum_{k=\ell}^{2\ell}\binom{m-\ell}{m-k}\binom{m+k}{2k}\binom{2m}{2k}^{-1} \times \frac{2\ell+1-k}{2^{m-k}} < 1. \tag{1.13}$$

Fact 2. The inequality (1.13) implies Theorem 1.2.

In [2], the proof of (1.13) is divided into two parts: first

Theorem 1.3. *For fixed $m \in \mathbb{N}$ and $0 \leq \ell < \left\lfloor \frac{m}{2} \right\rfloor$, the sum $S_{m,\ell}$ is increasing in ℓ.*

and then

Theorem 1.4. *The maximal sum $S_{m,\left\lfloor \frac{m-1}{2}\right\rfloor}$ is strictly less than 1. For m even, the maximal sum $S_{2m,m-1}$ is given by*

$$T(m) := S_{2m,m-1} = \sum_{r=2}^{m+1}\binom{2r}{r}\binom{m+1}{r}\frac{(r-1)}{2^r\binom{4m}{r}}. \tag{1.14}$$

a similar expression exists for m odd.

Fact 3. Theorems 1.3 and 1.4 imply Theorem 1.2.

These two results were established in [2] by some elementary estimates. These were long and do not extend to, for instance, the proof of logconcavity of $\{d_\ell(m)\}$. The hope is that the techniques used to provide the new proof of unimodality presented here, will also apply to other situations.

Section 2 presents a new elementary proof of Theorem 1.4 and Section 3 contains a proof based on a hypergeometric representation of $T(m)$. Section 4 shows that $T(m)$ converges to the value

$$\lim_{m\to\infty}\sum_{r=2}^{m+1}\binom{2r}{r}\binom{m+1}{r}\frac{(r-1)}{2^r\binom{4m}{r}} = \frac{2-\sqrt{2}}{2} \sim 0.292893. \tag{1.15}$$

This limit was *incorrectly* conjectured in [2] to be $1-\ln 2 \sim 0.306853$. The authors have failed to produce a proof of Theorem 1.3 by the automatic techniques developed in [11]. These methods yield recurrences for the summands in (1.13), but it is not possible to conclude from them that $S_{m,\ell}$ is increasing. The last section shows that the sequence $\{T(m) : m \geq 2\}$ is an increasing sequence.

2. The bound on $T(m)$

The result stated in Theorem 1.4 is equivalent to the bound

$$T(m) := \sum_{r=2}^{m+1} \binom{2r}{r}\binom{m+1}{r}\frac{(r-1)}{2^r\binom{4m}{r}} < 1. \tag{2.1}$$

A direct proof of this result is given next. Section 3 presents a proof based on a hypergeometric representation of $T(m)$.

Theorem 2.1. *The inequality $T(m) < 1$ holds.*

Proof. First, it is shown by induction that for m fixed and $2 \leq r \leq m+1$

$$a_m(r) := \binom{2r}{r}\binom{m+1}{r} \leq b_m(r) := \binom{4m}{r}. \tag{2.2}$$

If $r = 2$: $b_m(2) - a_m(2) = 5m(m-1) \geq 0$. Now observe that

$$\frac{b_m(r+1)}{b_m(r)} - \frac{a_m(r+1)}{a_m(r)} = \frac{4m-r}{r+1} - \frac{2(2r+1)(m+1-r)}{(r+1)^2} = \frac{2(m-1)+3r(r-1)}{(r+1)^2} > 0.$$

This gives the inductive step written as

$$b_m(r)\frac{b_m(r+1)}{b_m(r)} > a_m(r)\frac{a_m(r+1)}{a_m(r)}.$$

The inequality $a_m(r) < b_m(r)$ now yields

$$T(m) = \sum_{r=2}^{m+1} \frac{a_m(r)}{b_m(r)}\frac{r-1}{2^r} < \sum_{r=2}^{m+1} \frac{r-1}{2^r} = 1 - \frac{m+2}{2^{m+1}} < 1.$$

□

3. The hypergeometric representation of $T(m)$

This section provides a hypergeometric representation of the sum

$$T(m) = \sum_{r=2}^{m+1} \binom{2r}{r}\binom{m+1}{r}\frac{(r-1)}{2^r\binom{4m}{r}}. \tag{3.1}$$

Proposition 3.1. *The sum $T(m)$ is given by*

$$T(m) = 1 - {}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, -1-m\\ -4m\end{matrix}\right| 2\right) + \frac{m+1}{4m}\,{}_2F_1\left(\left.\begin{matrix}\frac{3}{2}, -m\\ 1-4m\end{matrix}\right| 2\right). \tag{3.2}$$

Proof. Since $\binom{m}{k} = \frac{(-1)^k(-m)_k}{k!}$, it follows that $\frac{\binom{m+1}{r}}{\binom{4m}{r}} = \frac{(-1-m)_r}{(-4m)_r}$. This relation and $\left(\frac{1}{2}\right)_r = (2r)!/(2^{2r}\,r!)$ give

$$T(m) = \sum_{r=2}^{m+1} \frac{\left(\frac{1}{2}\right)_r}{r!}\frac{(r-1)2^r(-1-m)_r}{(-4m)_r}. \tag{3.3}$$

Therefore

$$\begin{aligned}
T(m) &= -\sum_{r=2}^{m+1} \frac{\left(\frac{1}{2}\right)_r (-1-m)_r 2^r}{(-4m)_r \, r!} + \sum_{r=2}^{m+1} \frac{\left(\frac{1}{2}\right)_r (-1-m)_r 2^r}{(r-1)! \, (-4m)_r} \\
&= 1 + \frac{m+1}{4r} - \sum_{r=0}^{m+1} \frac{\left(\frac{1}{2}\right)_r (-1-m)_r 2^r}{(-4m)_r \, r!} + \frac{m+1}{4m} \sum_{r=0}^{m+1} \frac{\left(\frac{1}{2}\right)_r (-1-m)_r 2^r}{(-4m)_r \, (r-1)!} \frac{4m}{m+1} \\
&= 1 - \sum_{r=0}^{m+1} \frac{\left(\frac{1}{2}\right)_r (-1-m)_r}{(-4m)_r} \frac{2^r}{r!} + \frac{m+1}{4m} \left\{ 1 + \sum_{r=2}^{m+1} \frac{\left(\frac{1}{2}\right)_r 2^r}{(r-1)!} \frac{4m}{m+1} \frac{(-1-4m)_r}{(-4m)_r} \right\} \\
&= 1 - {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) + \frac{m+1}{4m} \sum_{r=2}^{m+1} \frac{\left(\frac{1}{2}\right)_r 2^r}{(r-1)!} \frac{4m}{m+1} \frac{(-1-m)_r}{(-4m)_r} \\
&= 1 - {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) + \frac{m+1}{4m} \sum_{r=0}^{m} \frac{\left(\frac{3}{2}\right)_r}{r!} \frac{2^r (-m)_r}{(1-4m)_r} \\
&= 1 - {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) + \frac{m+1}{4m} {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m \\ 1-4m \end{matrix} \right| 2 \right).
\end{aligned}$$

□

The next result provides an integral representation for $T(m)$.

Proposition 3.2. *The sum $T(m)$ is given by*

$$T(m) = \frac{3(m+1)}{16(4m-1)} \int_0^2 t \, {}_2F_1\left(\left. \begin{matrix} \frac{5}{2}, 1-m \\ 2-4m \end{matrix} \right| t \right) dt. \tag{3.4}$$

Proof. Integrate by parts and use

$$\frac{d}{dt} {}_2F_1\left(\left. \begin{matrix} a, b \\ c \end{matrix} \right| t \right) = \frac{ab}{c} {}_2F_1\left(\left. \begin{matrix} a+1, b+1 \\ c+1 \end{matrix} \right| t \right) \tag{3.5}$$

to produce

$$\int_0^2 t \, {}_2F_1\left(\left. \begin{matrix} \frac{5}{2}, 1-m \\ 2-4m \end{matrix} \right| t \right) dt = \frac{4(4m-1)}{3m} {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m \\ 1-4m \end{matrix} \right| 2 \right) - \frac{2(4m-1)}{3m} \int_0^2 {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m \\ 1-4m \end{matrix} \right| t \right) dt.$$

The last integral is evaluated using (3.5) to write

$${}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m \\ 1-4m \end{matrix} \right| t \right) = \frac{8m}{m+1} \frac{d}{dt} {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \right| t \right)$$

and the result follows. □

The next result provides a bound for the integrand in Proposition 3.2.

Proposition 3.3. *Let $n \in \mathbb{N}$, $n \geq 2$ and $0 \leq t \leq 2$. Then*

$$\left| {}_2F_1\left(\left. \begin{matrix} \frac{5}{2}, 1-m \\ 2-4m \end{matrix} \right| t \right) \right| \leq 9\sqrt{3}(3-t)^{-5/2}.$$

Proof. The hypergeometric function is given by

$${}_2F_1\left(\left. \begin{matrix} \frac{5}{2}, 1-m \\ 2-4m \end{matrix} \right| t \right) = \sum_{k=0}^{m-1} \left(\frac{5}{2} \right)_k \frac{(1-m)_k}{(2-4m)_k}.$$

The bound

$$\frac{(1-m)_k}{(2-4m)_k} \leq \frac{1}{3^k} \tag{3.6}$$

follows directly from the observation that $b_k(m) = 3^k(1-m)_k/(2-4m)_k$ satisfies $b_0(m) = 1$ and it is decreasing in k. Indeed,

$$\frac{b_{k+1}(m)}{b_k(m)} = \frac{3(1-m+k)}{2-4m+k} < 1. \tag{3.7}$$

Then (3.6) gives

$$\begin{aligned} {}_2F_1\left(\left. \begin{matrix} \frac{5}{2}, 1-m \\ 2-4m \end{matrix} \right| t \right) &\leq \sum_{k=0}^{m-1} \left(\frac{5}{2}\right)_k \frac{t^k}{3^k k!} \\ &\leq \sum_{k=0}^{\infty} \left(\frac{5}{2}\right)_k \frac{(t/3)^k}{k!} \\ &= {}_1F_0\left(\left. \begin{matrix} \frac{5}{2} \\ - \end{matrix} \right| \frac{t}{3} \right). \end{aligned}$$

The evaluation of the final hypergeometric sum comes from the binomial theorem

$${}_1F_0\left(\left. \begin{matrix} a \\ - \end{matrix} \right| z \right) = (1-z)^{-a}, \text{ for } |z| < 1. \tag{3.8}$$

□

The bound in Theorem 1.4 is now obtained.

Corollary 3.4. *For $m \in \mathbb{N}$, the function $T(m)$ satisfies*

$$T(m) < 1. \tag{3.9}$$

Proof. It is easy to compute that $T(1) = \frac{1}{4}$. For $m \geq 2$, observe that

$$\frac{3(m+1)}{16(4m-1)} = \frac{3}{16}\left(\frac{1}{4} + \frac{5/4}{4m-1}\right) \leq \frac{9}{112} \tag{3.10}$$

and thus

$$T(m) \leq \frac{9}{112} \int_0^2 \frac{9\sqrt{3}\, t\, dt}{(3-t)^{5/2}} = \frac{27}{28} < 1. \tag{3.11}$$

□

Note 3.5. This inequality completes the proof that $\{d_\ell(m)\}$ is unimodal.

4. The limiting behavior of $T(m)$

This section is devoted to establish the limiting value of $T(m)$.

Theorem 4.1. *The function $T(m)$ satisfies*

$$\lim_{m \to \infty} T(m) = \frac{2-\sqrt{2}}{2}. \tag{4.1}$$

The arguments will employ the classical Tannery theorem. This is stated next, a proof appears in [4], page 136.

Theorem 4.2. *(Tannery) Assume* $a_k := \lim_{m\to\infty} a_k(m)$ *satisfies* $|a_k(m)| \leq M_k$ *with* $\sum_{k=0}^{\infty} M_k < \infty$. *Then* $\lim_{m\to\infty} \sum_{k=0}^{m} a_k(m) = \sum_{k=0}^{\infty} a_k$.

Three proofs of Theorem 4.1 are presented here. In each one of them, the argument boils down to an exchange of limits. The first one is based on the integral representation of $T(m)$ and it uses bounded convergence theorem and Tannery's theorem. The second one deals directly with the hypergeometric sums and it employs Tannery's theorem for passing to the limit in a series. A similar argument can be employed in the third proof.

Proposition 4.3. *Assume* $0 \leq t < 4$ *is fixed. Then*

$$\lim_{m\to\infty} {}_2F_1\left(\left. \begin{matrix} \frac{5}{2}, 1-m \\ 2-4m \end{matrix} \right| t \right) = {}_1F_0\left(\left. \begin{matrix} \frac{5}{2} \\ - \end{matrix} \right| \frac{t}{4} \right) = \frac{32}{(4-t)^{5/2}}. \tag{4.2}$$

Proof. Start with

$${}_2F_1\left(\left. \begin{matrix} \frac{5}{2}, 1-m \\ 2-4m \end{matrix} \right| t \right) = \sum_{k=0}^{m-1} \frac{\left(\frac{5}{2}\right)_k (1-m)_k}{(2-4m)_k} \frac{t^k}{k!} \tag{4.3}$$

and observe that

$$\frac{(1-m)_k}{(2-4m)_k} = \prod_{j=0}^{k-1} \frac{m-1-j}{4m-2-j} \to \frac{1}{4^k} \tag{4.4}$$

as $m \to \infty$. Therefore

$$\lim_{m\to\infty} {}_2F_1\left(\left. \begin{matrix} \frac{5}{2}, 1-m \\ 2-4m \end{matrix} \right| t \right) = \sum_{k=0}^{\infty} \frac{\left(\frac{5}{2}\right)_k}{k!} \left(\frac{t}{4}\right)^k = {}_1F_0\left(\left. \begin{matrix} \frac{5}{2} \\ - \end{matrix} \right| \frac{t}{4} \right). \tag{4.5}$$

The hypergeometric sum is now evaluated using (3.8). □

The passage to the limit in (4.5) uses the Tannery's theorem. In this case

$$a_k(m) = \frac{\left(\frac{5}{2}\right)_k (1-m)_k}{(2-4m)_k} \frac{t^k}{k!} \tag{4.6}$$

satisfies

$$\begin{aligned} \lim_{m\to\infty} a_k(m) &= \lim_{m\to\infty} \left(\frac{5}{2}\right)_k \frac{t^k}{k!} \frac{\left(\frac{1}{m}-1\right)\left(\frac{2}{m}-1\right)\cdots\left(\frac{k}{m}-1\right)}{\left(\frac{2}{m}-4\right)\left(\frac{3}{m}-1\right)\cdots\left(\frac{1+k}{m}-4\right)} \\ &= \left(\frac{5}{2}\right)_k \frac{t^k}{k!\,4^k} \end{aligned}$$

exists. This limit is denoted by a_k.

The result now follows from the bound

$$|a_k(m)| \leq M_k := \left(\frac{5}{2}\right)_k \frac{t^k}{k!\,3^k}, \tag{4.7}$$

and the sum

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} \left(\frac{5}{2}\right)_k \frac{t^k}{k!\,3^k} = \left(1 - \frac{t}{3}\right)^{-5/2}. \tag{4.8}$$

valid for $0 \leq t \leq 2$. Tannery's theorem gives

$$\lim_{m\to\infty} \sum_{k=0}^{m-1} a_k(m) = \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \left(\frac{5}{2}\right)_k \frac{t^k}{k!\, 4^k} = \left(1 - \frac{t}{4}\right)^{-5/2}. \tag{4.9}$$

The expression in Proposition 3.2, the bound (3.6) and Proposition 3.3 give, via the dominated convergence theorem, the value

$$\begin{aligned} \lim_{m\to\infty} T(m) &= \lim_{m\to\infty} \frac{3(m+1)}{16(4m-1)} \int_0^2 {}_2F_1\left(\left. \begin{matrix} \frac{5}{2}, 1-m \\ 2-4m \end{matrix} \right| t \right) t\, dt \\ &= \frac{3}{64} \int_0^2 {}_1F_0\left(\left. \begin{matrix} \frac{5}{2} \\ - \end{matrix} \right| \frac{t}{4} \right) dt \\ &= \frac{3}{64} \int_0^2 \frac{32t}{(4-t)^{5/2}}\, dt \\ &= \frac{2-\sqrt{2}}{2}. \end{aligned} \tag{4.10}$$

This completes the first proof.

The second proof of the limiting value of $T(m)$ uses the hypergeometric representation of $T(m)$ in (3.2). It amounts to proving

$$\lim_{m\to\infty} {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) - \frac{m+1}{4m} {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m \\ 1-4m \end{matrix} \right| 2 \right) = \frac{\sqrt{2}}{2}. \tag{4.11}$$

The contiguous relation [12], page 28,

$${}_2F_1\left(\left. \begin{matrix} a+1, b \\ c \end{matrix} \right| z \right) = {}_2F_1\left(\left. \begin{matrix} a, b \\ c \end{matrix} \right| z \right) + \frac{bz}{c} {}_2F_1\left(\left. \begin{matrix} a+1, b+1 \\ c+1 \end{matrix} \right| z \right) \tag{4.12}$$

is used with $a = \frac{1}{2}$, $b = -1-m$, $c = -4m$ and $z = 2$ to obtain

$${}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) = {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) + \frac{m+1}{2m} {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m \\ 1-4m \end{matrix} \right| 2 \right)$$

and this gives

$$\frac{(m+1)}{4m} {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m \\ 1-4m \end{matrix} \right| 2 \right) = \frac{1}{2} \left({}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) - {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) \right). \tag{4.13}$$

Thus if suffices to prove

$$\lim_{m\to\infty} 3\, {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) - {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) = \sqrt{2}. \tag{4.14}$$

A direct calculation shows that

$$3\, {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) - {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -1-m \\ -4m \end{matrix} \right| 2 \right) = \sum_{k=0}^{m+1} a_k(m)$$

with

$$a_k(m) = \sum_{k=0}^{m+1} \frac{\left[3\left(\frac{1}{2}\right)_k - \left(\frac{3}{2}\right)_k \right] (-1-m)_k 2^k}{(-4m)_k\, k!}. \tag{4.15}$$

The question is now reduced to justifying passing to the limit in

$$\lim_{m\to\infty} \sum_{k=0}^{m+1} a_k(m) = \sum_{k=0}^{\infty} \lim_{m\to\infty} a_k(m) \tag{4.16}$$

since

$$\lim_{m\to\infty} a_k(m) = \left(3\,\left(\tfrac{1}{2}\right)_k - \left(\tfrac{3}{2}\right)_k\right) \frac{1}{k!\,2^k} \tag{4.17}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} \lim_{m\to\infty} a_k(m) &= \sum_{k=0}^{\infty} \left(3\,\left(\tfrac{1}{2}\right)_k - \left(\tfrac{3}{2}\right)_k\right) \frac{1}{k!\,2^k} \\
&= 3\sum_{k=0}^{\infty} \frac{\left(\tfrac{1}{2}\right)_k 2^{-k}}{k!} - \sum_{k=0}^{\infty} \frac{\left(\tfrac{3}{2}\right)_k 2^{-k}}{k!} \\
&= 3(1-1/2)^{-1/2} - (1-1/2)^{-3/2} \\
&= \sqrt{2}.
\end{aligned}$$

The last step is justified using Tannery's theorem. In the present case $a_k(m)$, given in (4.15), satisfies

$$|a_k(m)| \leq \left(3\left(\tfrac{1}{2}\right)_k + \left(\tfrac{3}{2}\right)_k\right) \frac{2^k}{k!} \frac{(-1-m)_k}{(-4m)_k}. \tag{4.18}$$

The proof of the inequality

$$\frac{(-1-m)_k}{(-4m)_k} \leq \frac{1}{3^k}, \tag{4.19}$$

is similar to the proof of (3.6). This is then used to verify that the hypothesis of Tannery's theorem are satisfied. The details are omitted.

A third proof is based on the analysis of a function that resembles the formula for $T(m)$.

Proposition 4.4. *For* $0 \leq x < 1$ *define*

$$W_m(x) = \sum_{r=0}^{m+1} \binom{2r}{r}\binom{m+1}{r}\binom{4m}{r}^{-1} x^r. \tag{4.20}$$

Then

$$\lim_{m\to\infty} W_m(x) = \frac{1}{\sqrt{1-x}} \quad \text{and} \quad \lim_{m\to\infty} x\frac{d}{dx}W_m(x) = \frac{x}{2(1-x)^{3/2}}. \tag{4.21}$$

Proof. Note that the sum defining $W_m(x)$ can be extended to infinity since $\binom{m+1}{r}$ has compact support. The proof now follows from

$$W_m(x) = \sum_{r=0}^{\infty} \binom{2r}{r}\left(\frac{x}{4}\right)^r \prod_{i=1}^{r} \left(\frac{1-\frac{i-2}{m}}{1-\frac{i-1}{m}}\right) \to \sum_{r=0}^{\infty} \binom{2r}{r}\left(\frac{x}{4}\right)^r = \frac{1}{\sqrt{1-x}},$$

where the passage to the uniform limit is justified by Weierstrass M-test or dominated convergence theorem. The second assertion is immediate. □

Corollary 4.5. *The sum $T(m)$ satisfies*

$$\lim_{m\to\infty} T(m) = \frac{2-\sqrt{2}}{\sqrt{2}}. \tag{4.22}$$

Proof. This follows from the identity

$$T(m) = \lim_{x\to 1/2} \frac{1}{2}\frac{d}{dx}W_m(x) - W_m(x). \tag{4.23}$$

□

Note 4.6. The function $W_m(x)$ can be expressed in hypergeometric form as

$$W_m(x) = {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \right| 4x \right). \tag{4.24}$$

5. The monotonicity of $T(m)$

This last section describes the convergence of $T(m)$ to its limit given in (4.1).

Theorem 5.1. *The function $T(m)$ is monotone increasing for $m \geq 2$.*

Proof. Let

$$F(r,m) = \binom{2r}{r}\binom{m+1}{r}\frac{r-1}{2^r\binom{4m}{r}}. \tag{5.1}$$

The proof is based on a recurrence involving $F(r,m)$ that is obtained by the WZ-technology as developed in [11]. Input the hypergeometric function $F(k,m)$ into WZ-package with summing range from $r=2$ to $r=n+1$. The recurrence relations that come as the ouput is

$$a(n)T(n) - b(n)T(n+1) + c(n)T(n+2) + d(n) = 0, \tag{5.2}$$

where

$$\begin{aligned}
a(n) = {} & 7195230 + 87693273n + 448856568n^2 + 1263033897n^3 + 2147597568n^4 \\
& +2279791176n^5 + 1502157312n^6 + 586779648n^7 + 121208832n^8 + 9732096n^9 \\
b(n) = {} & 9661680 + 123557904n + 651005760n^2 + 1865031680n^3 + 3206772480n^4 \\
& +3428727552n^5 + 2272235520n^6 + 894167040n^7 + 187269120n^8 + 15499264n^9 \\
c(n) = {} & 3265920 + 41472576n + 217055232n^2 + 618806528n^3 + 1062162432n^4 \\
& +1139030016n^5 + 762052608n^6 + 305528832n^7 + 66060288n^8 + 5767168n^9 \\
d(n) = {} & -799470 - 5607945n - 14906040n^2 - 16808745n^3 - 2987520n^4 + 9906360n^5 \\
& +8025600n^6 + 1858560n^7.
\end{aligned}$$

Note that $b(n) = a(n) + c(n) + d(n)$, then (5.2) becomes

$$a(n)T(n) - (a(n)+c(n)+d(n))T(n+1) + c(n)T(n+2) + d(n) = 0, \tag{5.3}$$

which is written as

$$a(n)(T(n)-T(n+1)) + d(n)(1-T(n+1)) = c(n)(T(n+1)-T(n+2)). \tag{5.4}$$

Lemma 5.2. *The polynomial $d(m)$ is nonnegative for $m \geq 2$.*

Proof. Simply observe that

$$\begin{aligned}
d(x+2) = {} & 814627800 + 2803521195x + 3780146130x^2 + 2680435095x^3 \\
& 1098008880x^4 + 262332600x^5 + 34045440x^6 + 1858560x^7
\end{aligned}$$

is a polynomial with positive coefficients. □

Theorem 2.1 shows that $T(m) < 1$ and with this Lemma 5.2 implies

$$a(n)(T(n) - T(n+1)) \leq c(n)(T(n+1) - T(n+2)). \tag{5.5}$$

Assume T is not monotone. Define N as the smallest positive integer such that

$$T(N) > T(N+1). \tag{5.6}$$

Then (5.5) implies

$$a(N)(T(N) - T(N+1)) \leq c(N)(T(N+1) - T(N+2)) \tag{5.7}$$

and since $a(N) > 0$, $c(N) > 0$, it follows that $T(N+1) > T(N+2)$. Iteration of this argument shows that the sequence $\{T(n) : n \geq N\}$ is monotonically decreasing.

Let $\delta_N = T(N) - T(N+1) > 0$, then (5.7) yields

$$T(N+1) - T(N+2) \geq \frac{a(N)}{c(N)} \delta_N. \tag{5.8}$$

Iterating this procedure gives

$$T(N+p) - T(N+p+1) > \delta_N \prod_{i=0}^{p-1} \frac{a(N+i)}{c(N+i)}, \text{ for every } p \in \mathbb{N}. \tag{5.9}$$

This inequality is now impossible as $p \to \infty$, since the left-hand side converges to 0 in view of (4.1) and

$$\lim_{n \to \infty} \frac{a_n}{c_n} = \frac{27}{16} \tag{5.10}$$

showing that the right-hand side blows up. □

6. A conjectured inequality for hypergeometric functions

The hypergeometric representation for the function $T(m)$ and the monotonicity of $T(m)$ give using (4.13),

$$ {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m-2 \\ -4m-4 \end{matrix} \right| 2 \right) - {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m-1 \\ -4m \end{matrix} \right| 2 \right) > 3\left[{}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -m-2 \\ -4m-4 \end{matrix} \right| 2 \right) - {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -m-1 \\ -4m \end{matrix} \right| 2 \right) \right]. $$

This is the special case $x = \frac{1}{2}$ of the conjecture given below.

Conjecture 6.1. *The inequality*

$$ {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m-2 \\ -4m-4 \end{matrix} \right| 4x \right) - {}_2F_1\left(\left. \begin{matrix} \frac{3}{2}, -m-1 \\ -4m \end{matrix} \right| 4x \right) > 3\left[{}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -m-2 \\ -4m-4 \end{matrix} \right| 4x \right) - {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, -m-1 \\ -4m \end{matrix} \right| 4x \right) \right] $$

holds for $x \geq \frac{1}{2}$.

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