ZEROS OF COMBINATIONS OF THE RIEMANN ξ-FUNCTION ON BOUNDED VERTICAL SHIFTS

ATUL DIXIT, NICOLAS ROBLES, ARINDAM ROY, AND ALEXANDRU ZAHARESCU

Dedicated to G. H. Hardy on the centenary of his 1914 *article proving the infinitude of the zeros of* $\zeta(s)$ *on the critical line.*

Abstract. In this paper we consider a series of bounded vertical shifts of the Riemann ξ-function. Interestingly, although such functions have essential singularities, infinitely many of their zeros lie on the critical line. We also generalize some integral identities associated with the theta transformation formula and some formulae of G. H. Hardy and W. L. Ferrar in the context of a pair of functions reciprocal in Fourier cosine transform.

1. Introduction

The study of the zeros and the 'a-points' of the Riemann zeta-function is of special interest. It is more difficult to locate the zeros or the 'a-points' than to study the value distributions of $\zeta(s)$.

The behavior of $\zeta(s)$ on every vertical line $\sigma = \mathrm{Re}(s) > \frac{1}{2}$ has been studied by Bohr and his collaborators. Let us take the half-plane $\sigma > \frac{1}{2}$, and remove all the points which have the same imaginary part as, and smaller real part than, one of the possible zeros (or the pole) of $\zeta(s)$ in this region. We denote the remaining part of this perforated half-plane by $\mathcal{G}$. Specifically, Bohr and Jessen [3, 4] discovered that for $\sigma > \frac{1}{2}$, the limit

$$\lim_{T\to\infty} \frac{1}{T}\mu\{\tau \in [0,T] : \sigma + i\tau \in \mathcal{G}, \log\zeta(\sigma + i\tau) \in \mathcal{R}\}$$

exists. Here μ is the Lebesgue measure and $\mathcal{R}$ is any fixed rectangle whose sides are parallel to the axes. Later Voronin [28] provided a generalization of Bohr's denseness result.

> *For any fixed and distinct numbers* $s_1, s_2, \ldots, s_n$ *with* $\frac{1}{2} < \mathrm{Re}(s_k) < 1$*, the set* $\{(\zeta(s_1 + it), \ldots, \zeta(s_n + it)) : t \in \mathbb{R}\}$ *is dense in* $\mathbb{C}^n$*. Moreover, for any* s *with* $\frac{1}{2} < \mathrm{Re}(s) < 1$*, the set* $\{(\zeta(s + it), \ldots, \zeta^{(n)}(s + it)) : t \in \mathbb{R}\}$ *is dense in* $\mathbb{C}^n$*.*

Even more striking is Voronin's [29] universality theorem.

> *Let* $0 < r < \frac{1}{4}$ *and* $g(s)$ *be a non-zero analytic function on* $|s| \le r$*. Then for any* $\epsilon > 0$*, there exists a positive real number* τ *such that*
>
> $$\max_{|s|\le r}\left|\zeta\left(s + \frac{3}{4} + i\tau\right) - g(s)\right| < \epsilon.$$
>
> *Moreover,*
>
> $$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{|s|\le r}\left|\zeta\left(s + \frac{3}{4} + i\tau\right) - g(s)\right| < \epsilon\right\} > 0.$$

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Concerning Voronin's theorem, Bagchi [2] gave an equivalent condition for the Riemann hypothesis for $\zeta(s)$. He proved in his doctoral thesis that

The Riemann hypothesis is true if, and only if, for any $\epsilon > 0$

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{|s|\leq r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-\zeta(s)\right| < \epsilon\right\} > 0.$$

These connections motivate us to study the vertical shifts $s \to s + i\tau$ of $\zeta(s)$. The values of $\zeta(s)$ on certain vertical arithmetic progressions have been studied by various authors. Putnam [20, 21] showed that for any $d > 0$, the sequence $d, 2d, 3d, \cdots$ contains an infinity of elements which are not the imaginary parts of zeros of $\zeta(s)$ on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. More recently, van Frankenhuijesen [27] obtained bounds for the length of any hypothetical arithmetic progression. Good [8] found asymptotic formulae for the discrete second and fourth moments of $\zeta(s)$ on arbitrary arithmetic progressions to the right of the critical line. Steuding and Wegert [24] succeeded in obtaining an asymptotic formula for the discrete first moment of $\zeta(s)$ on arbitrary arithmetic progressions in the critical strip. Li and Radziwiłł [18] established, among many other things, results on the distribution of values of $\zeta(\frac{1}{2}+i(al+b))$ as l ranges over the integers in some dyadic interval $[T, 2T]$.

Set

$$\eta(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad \text{and} \quad \rho(t) := \eta\left(\frac{1}{2}+it\right).$$

It is well-known that $\eta(s)$ is a meromorphic function with poles at $s = 0$ and 1, and that $\rho(t)$ is a real-valued function. Hardy [10] proved that $\rho(t)$ has infinitely many real zeros. In other words, the Riemann zeta-function has infinitely many zeros on the critical line.

Let $\{c_j\}$ be a sequence of real numbers such that

$$\sum_{j=1}^{\infty} |c_j| < \infty.$$

Let $\{\lambda_j\}$ be a bounded sequence of real numbers. Define

$$F(s) := \sum_{j=1}^{\infty} c_j \eta(s+i\lambda_j). \tag{1.1}$$

Note that $F(s)$ has poles at $-i\lambda_j$ and $1-i\lambda_j$ for all j. Using Stirling's formula for the gamma function (see (2.5) below) and the boundedness of the sequence $\{\lambda_j\}$, one can show that there exists a bounded set $D \subset \mathbb{C}$ such that $F(s)$ is analytic on $\mathbb{C} \setminus D$. In particular, D can be taken as the union of two bounded vertical intervals containing the points 0 and 1. If $\{\lambda_j\}$ is an infinite sequence, then $F(s)$ has essential singularities inside the set D. Independently of whether $F(s)$ has essential singularities or not, it can be seen that $F(\frac{1}{2}+it)$ is a well-defined, real-valued function for $t \in \mathbb{R}$. A natural question arises - what can we say about the zeros of $F(s)$?

Without loss of generality, we can take $c_j \neq 0$ and the λ_j's to be distinct for all j. Indeed, the terms corresponding to c_j's being zero have no contribution to $F(s)$. Furthermore, since the right-hand side of (1.1) is absolutely convergent, the terms for which $\lambda_i = \lambda_j$ can be grouped together and we can denote the new coefficient by c_j. We have the following result.

Theorem 1.1. *Let $\{c_j\}$ be a sequence of non-zero real numbers so that $\sum_{j=1}^{\infty}|c_j| < \infty$. Let $\{\lambda_j\}$ be a bounded sequence of distinct real numbers that attains its bounds. Then the function $F(s) = \sum_{j=1}^{\infty} c_j\eta(s+i\lambda_j)$ has infinitely many zeros on the critical line* $\mathrm{Re}(s) = \frac{1}{2}$.

Let

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad \text{and} \quad \Xi(t) := \xi\left(\frac{1}{2}+it\right).$$

One of the essential ingredients in the proof of Theorem 1.1 is the integral identity

$$\int_0^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}}\cos(\nu t)dt = \frac{\pi}{2}(e^{\frac{\nu}{2}} - 2e^{-\frac{\nu}{2}}\hat{\vartheta}_3(e^{-2\nu})), \tag{1.2}$$

which was used by Hardy as well to prove that the Riemann zeta-function has infinitely many zeros on the critical line (see [10], and §2.16 of [26]). Here

$$\hat{\vartheta}_3(x) := \sum_{n=1}^{\infty}\exp(-x\pi n^2). \tag{1.3}$$

Note that for $\mathrm{Re}(x) > 0$, the functional equation

$$2\hat{\vartheta}_3(x) + 1 = x^{-\frac{1}{2}}(2\hat{\vartheta}_3(x^{-1}) + 1) \tag{1.4}$$

holds.

Identities such as (1.2) have many applications. For example, using Fourier's integral theorem, we can invert identities like (1.2) to obtain new expressions for the Riemann Ξ-function, and hence for the Riemann zeta-function. For example, if we replace ν by $-\nu$ in (1.2) and use Fourier's integral theorem, we have

$$\Xi(t) = (t^2+\tfrac{1}{4})\int_0^{\infty}(e^{-\frac{u}{2}} - e^{\frac{u}{2}}\hat{\vartheta}_3(e^{2u}))\cos(xu)\,du,$$

which is well-known [26, p. 254, Equation (10.1.1)]. New integral identities of the type (1.2), having the function $\Xi(t)$ under the integral sign, were studied by Ramanujan [23, Equations (12), (16)] (see also[12, p. 37]) and later by Koshliakov [15, Equations (18), (25) and (38)], [14, p. 404–405]. For recent work in this direction, see the survey article due to one of the authors [6]. At this juncture, it is important to note the following quote by Hardy [12]:

> ... *The unsolved problems concerning the zeros of $\zeta(s)$ or of $\Xi(t)$ are among the most obscure and difficult in the whole range of Pure Mathematics. Any new formulae involving $\zeta(s)$ and $\Xi(t)$ are of very great interest, because of the possibility that they may throw new light on some outstanding questions...*

Another goal of this paper is to generalize identity (1.2) and other allied identities by replacing the term $\cos(xt)$ in (1.2) by a more general class of functions that will be discussed shortly. These allied identities are similar in nature to (1.2) in that they involve integrals of the form

$$\int_0^{\infty} f\left(\tfrac{t}{2}\right)\Xi\left(\tfrac{t}{2}\right)\cos(\tfrac{1}{2}t\log\alpha)\,dt,$$

where $f(t)$ is of the form $f(t) = g(it)g(-it)$ with g analytic in t. Two well-known examples of such identities are those due to Ferrar [7] and Hardy [11, Equation (2)]. In this paper, these two identities are derived as special cases of more general results given below (see Corollaries 1.3 and 1.4 of Theorems 1.3 and 1.4 respectively).

Let $\phi(x)$ and $\psi(x)$ be two functions integrable on $\mathbb{R}$. The functions ϕ and ψ are said to be reciprocal in Fourier cosine transform if

$$\frac{\sqrt{\pi}}{2}\phi(x)=\int_0^\infty \psi(u)\cos(2ux)du \quad \text{and} \quad \frac{\sqrt{\pi}}{2}\psi(x)=\int_0^\infty \phi(u)\cos(2ux)du. \tag{1.5}$$

We define $Z_1(s)$ and $Z_2(s)$ in terms of the Mellin transforms of ϕ and ψ by

$$\Gamma\left(\frac{s}{2}\right)Z_1(s):=\int_0^\infty x^{s-1}\phi(x)dx, \quad \Gamma\left(\frac{s}{2}\right)Z_2(s):=\int_0^\infty x^{s-1}\psi(x)dx, \tag{1.6}$$

each valid in a specific vertical strip in the complex s-plane. Throughout this paper, by $\int_{(c)}$ we mean $\int_{c-i\infty}^{c+i\infty}$. Note that in case of a non-empty intersection of the two corresponding vertical strips, the Mellin inversion theorem gives

$$\phi(x)=\frac{1}{2\pi i}\int_{(c)}\Gamma\left(\frac{s}{2}\right)Z_1(s)x^{-s}ds, \quad \psi(x)=\frac{1}{2\pi i}\int_{(c)}\Gamma\left(\frac{s}{2}\right)Z_2(s)x^{-s}ds, \tag{1.7}$$

where $\mathrm{Re}(s)=c$ lies in the intersection. Moreover, let us define

$$\Theta(x):=\phi(x)+\psi(x) \quad \text{and} \quad Z(s):=Z_1(s)+Z_2(s) \tag{1.8}$$

so that

$$\Gamma\left(\frac{s}{2}\right)Z(s)=\int_0^\infty x^{s-1}\Theta(x)dx \tag{1.9}$$

for values of s in the intersection of the two strips.

Define a class K of functions as follows (see also [16]):

Definition 1.1. Let $0<\omega\leq\pi$ and $\lambda<\frac{1}{2}$. If $f(z)$ is such that

i) $f(z)$ is analytic with $z=re^{i\theta}$, regular in the angle defined by $r>0$, $|\theta|<\omega$,
ii) $f(z)$ satisfies the bounds

$$f(z)=\begin{cases} O(|z|^{-\lambda-\varepsilon}) & \text{if } |z| \text{ is small,} \\ O(|z|^{-\beta-\varepsilon}) & \text{if } |z| \text{ is large,} \end{cases} \tag{1.10}$$

for every $\varepsilon>0$ and $\beta>\lambda$, and uniformly in any angle $\theta<\omega$,

then we say that f belongs to the class K and write $f(z)\in K(\omega,\lambda,\beta)$.

Our next results are as follows.

Theorem 1.2. *Let $\beta>1$ and $\phi,\psi\in K(\omega,0,\beta)$ and suppose that Θ and Z are defined as in* (1.8)*. Then we have*

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}Z\left(\frac{1}{2}+it\right)dt=\frac{\pi}{2}Z(1)-\frac{\pi}{2}\sum_{n=1}^\infty \Theta(\pi^{1/2}n). \tag{1.11}$$

Corollary 1.1. *Identity* (1.2) *is a special case of Theorem* 1.2 *with* $\phi(x)=\psi(x)=\exp(-e^{2\nu}x^2)$ *for* $-\frac{\pi}{4}<\mathrm{Im}(\nu)<\frac{\pi}{4}$.

The following result, which was obtained by one of the authors in [5] to study a generalization of the identity (1.2), is also a special case of Theorem 1.2.

Corollary 1.2. *Let $h \in \mathbb{C}$ be fixed and x such that $-\frac{\pi}{4} < x < \frac{\pi}{4}$. If we set*

$$\varpi(h,x) := \sum_{n=1}^{\infty} \exp(-\pi x n^2)\cos(\pi^{1/2} h n), \tag{1.12}$$

and

$$\nabla(h,t,e^{-2x}) := e^{xit}{}_1F_1\left(\frac{1}{4}+\frac{1}{2}it; \frac{1}{2}; -\left(\frac{he^{x}}{2}\right)^2\right) + e^{-xit}{}_1F_1\left(\frac{1}{4}-\frac{1}{2}it; \frac{1}{2}; -\left(\frac{he^{x}}{2}\right)^2\right),$$

where ${}_1F_1$ is the confluent hypergeometric function, then

$$\int_0^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} \nabla(h,t,e^{-2x})dt = \pi(e^{x/2}e^{-h^2/(4e^{-2x})} - 2e^{-x/2}\varpi(h,e^{-2x})). \tag{1.13}$$

Our next result generalizes a formula due to Hardy [11, Equation (2)].

Theorem 1.3. *Let $\beta > 1$ and $\phi, \psi \in K(\omega, 0, \beta)$ and suppose that Θ and Z are defined as in (1.8). Then we have*

$$\int_0^{\infty} \frac{\Xi(\frac{t}{2})}{(1+t^2)} \frac{Z(\frac{1+it}{2})}{\cosh\frac{1}{2}\pi t}\, dt = \frac{-1}{4}\left\{\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{\Theta(x)}{x+n\sqrt{\pi}}\, dx - \frac{Z(1)}{n}\right) + Z'(1) + \frac{(\gamma-\log 4\pi)}{2}Z(1)\right\}. \tag{1.14}$$

Corollary 1.3. *Let $\psi(x)$ denote the logarithmic derivative of the gamma function $\Gamma(x)$. For $a > 0$,*

$$2\int_0^{\infty} \frac{\Xi(\frac{t}{2})}{1+t^2} \frac{\cos(\frac{1}{2}t\log a)}{\cosh\frac{1}{2}\pi t}\, dt = \sqrt{a}\int_0^{\infty} e^{-\pi a^2 x^2}(\psi(x+1)-\log x)\, dx. \tag{1.15}$$

Ferrar's formula [7] is generalized in this paper to the following.

Theorem 1.4. *Let $\beta > 1$ and $\phi, \psi \in K(\omega, 0, \beta)$ and suppose that Θ and Z are defined as in (1.8). Then we have*

$$\begin{aligned} &4\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\frac{\Xi(\frac{t}{2})}{1+t^2}Z\left(\frac{1}{2}+it\right)dt \\ &= -\pi^{3/2}\left\{2Z'(1) + (\gamma - \log 16\pi)Z(1) + 2\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{\Theta(x)}{\sqrt{x^2+\pi n^2}}\, dx - \frac{Z(1)}{n}\right)\right\}. \end{aligned} \tag{1.16}$$

Corollary 1.4. *Let $K_\nu(z)$ denote the modified Bessel function of order ν. For $a > 0$,*

$$\begin{aligned} &\frac{-1}{\sqrt{\pi}}\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\frac{\Xi(\frac{t}{2})}{1+t^2}\cos\left(\frac{1}{2}t\log a\right)dt \\ &= \sqrt{\alpha}\int_0^{\infty} e^{-\frac{a^2t^2}{4\pi}}\left(\sum_{n=1}^{\infty}K_0(nt) - \frac{\pi}{2t}\right)dt. \end{aligned} \tag{1.17}$$

This corollary is proved here using the following lemma, which is interesting in its own right, since it gives an example of a function self-reciprocal in Fourier cosine transform that seems to have been unnoticed before.

Lemma 1.5. *For $y > 0$, we have*

$$\int_0^\infty \left(\sum_{n=1}^\infty K_0(2\pi nx) - \frac{1}{4x} \right) \cos(2\pi yx)\, dx = \frac{1}{2} \left(\sum_{n=1}^\infty K_0(2\pi ny) - \frac{1}{4y} \right). \tag{1.18}$$

It is interesting to note here that Watson [30, p. 303] proved that for $\mathrm{Re}(\nu) > 0$, the function

$$\frac{1}{2}\Gamma(\nu) + 2\sum_{n=1}^\infty \left(nx\sqrt{\frac{\pi}{2}} \right)^\nu K_\nu(nx\sqrt{2\pi})$$

is self-reciprocal in the generalized Hankel transform of order $2\nu - \frac{1}{2}$.

This paper is organized as follows. In Section 2, we give many preliminary results all of which are subsequently used in the sequel. Section 3 is devoted to proving Theorem 1.1. We prove Theorem 1.2 and Corollaries 1.1 and 1.2 in Section 4. In Section 5, we prove Theorem 1.3 and Corollary 1.3. Finally in Section 6, we prove Theorem 1.4 and Corollary 1.4. We conclude the paper with a proof of Lemma 1.5.

2. Preliminaries

The following lemmas are instrumental in the proofs of our theorems.

Lemma 2.1. *For $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$,*

$$\int_{-\infty}^\infty e^{\alpha t} \rho(t)\, dt = -4\pi \cos\frac{\alpha}{2} + 2\pi e^{\frac{\alpha i}{2}} \sum_{n=-\infty}^\infty e^{-n^2\pi e^{2\alpha i}}. \tag{2.1}$$

For details see Landau [17, section 3] or Remark 4.1 below.

Lemma 2.2. *Let $h : \mathbb{C} \to \mathbb{C}$ be analytic at $\alpha = \frac{\pi}{4}$. As $\alpha \to \frac{\pi}{4}$, we have*

$$\frac{d^m}{d\alpha^m} \left(h(\alpha) \sum_{n=-\infty}^\infty e^{-n^2\pi e^{2\alpha i}} \right) \to 0. \tag{2.2}$$

The above result can be adapted from Landau [17, section 3] and Titchmarsh [26, p. 257]. For the sake of completeness, we supply the proof here.

Proof. Let $\hat{\vartheta}_3(\delta)$ be defined as in (1.3). Then $\hat{\vartheta}_3$ is analytic for $-\frac{\pi}{2} < \arg\delta < \frac{\pi}{2}$. Note that

$$\hat{\vartheta}_3(i+\delta) = \sum_{n=1}^\infty e^{-n^2\pi(i+\delta)} = \sum_{n=1}^\infty (-1)^n e^{-n^2\pi\delta}.$$

Next we have

$$\hat{\vartheta}_3(4\delta) = \sum_{n=1}^\infty e^{-4\delta n^2\pi} = \sum_{n=1}^\infty e^{-(2n)^2\pi\delta} = \sum_{m\in 2\mathbb{N}} e^{-m^2\pi\delta},$$

as well as

$$\hat{\vartheta}_3(\delta) = \sum_{m\in 2\mathbb{N}} e^{-m^2\pi\delta} + \sum_{m\in 2\mathbb{N}+1} e^{-m^2\pi\delta}.$$

Therefore,

$$2\hat{\vartheta}_3(4\delta) - \hat{\vartheta}_3(\delta) = \sum_{m\in 2\mathbb{N}} e^{-m^2\pi\delta} - \sum_{m\in 2\mathbb{N}+1} e^{-m^2\pi\delta} = \sum_{m\in\mathbb{N}} (-1)^m e^{-m^2\pi\delta} = \hat{\vartheta}_3(i+\delta).$$

Using (1.4), we have

$$\begin{aligned}\hat{\vartheta}_3(i+\delta) &= \frac{1}{\sqrt{\delta}}\hat{\vartheta}_3\left(\frac{1}{4\delta}\right) + \frac{1}{2\sqrt{\delta}} - 1 - \frac{1}{\sqrt{\delta}}\hat{\vartheta}_3\left(\frac{1}{\delta}\right) - \frac{1}{2}\frac{1}{\sqrt{\delta}} + \frac{1}{2}\\ &= \frac{1}{\sqrt{\delta}}\hat{\vartheta}_3\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\hat{\vartheta}_3\left(\frac{1}{\delta}\right) - \frac{1}{2}.\end{aligned}$$

Since $\exp(-1/x)$ tends to zero as $x \to 0$ faster than any power x^{-k} going to infinity (for $k > 0$), we see that as $\delta \to 0^+$, the function $\frac{1}{2} + \hat{\vartheta}_3(i+\delta)$ and all of its derivatives tend to zero. Since $h(\alpha)$ is analytic at $\pi/4$ and

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}} = 1 + 2\hat{\vartheta}_3(e^{2\alpha i}),$$

if $\alpha \to \frac{1}{4}\pi^-$, i.e., $e^{2i\alpha} \to i$ along any path in the wedge $|\arg(e^{2i\alpha} - i)| < \frac{1}{2}\pi$,

$$\lim_{\alpha \to \frac{1}{4}\pi^-} \frac{d^m}{d\alpha^m}[h(\alpha)(1 + 2\hat{\vartheta}_3(e^{2i\alpha}))] = 0.$$

This proves the lemma. □

The following result is due to Kronecker (see Hardy and Wright [13]).

Lemma 2.3. *Let $(n\theta)$ denote the fractional part of $n\theta$. If θ is irrational, then the set of points $(n\theta)$ is dense in the interval $(0,1)$.*

Remark 2.1. If θ is rational, then the set of points $(n\theta)$ is periodic in the interval $(0,1)$.

Two functions $\phi(x)$ and $\psi(x)$ are said to be reciprocal in the Hankel transformation of order ν if

$$\phi(x) = 2\int_0^\infty (ux)^{\frac{1}{2}} J_\nu(2ux)\psi(u)\,du \quad \text{and} \quad \psi(x) = 2\int_0^\infty (ux)^{\frac{1}{2}} J_\nu(2ux)\phi(u)\,du, \tag{2.3}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν defined by [1, p. 200, Equation (4.5.2)]

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}.$$

We now give below a result due to Kühn and two of the present authors [16, Lemma 2.2].

Lemma 2.4. *Let ϕ and ψ be reciprocal functions under the Hankel transformation of order ν defined in (2.3). Let $\phi, \psi \in K(\omega, \lambda, \beta)$. Then there exist two regular functions Φ and Ψ such that*

$$\begin{aligned}\phi(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{4}\right)\Phi(s)x^{-s}\,ds,\\ \psi(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{4}\right)\Psi(s)x^{-s}\,ds\end{aligned}$$

for $c > 0$. Moreover Φ and Ψ satisfy the following:

(1) $\Phi(s) = \Psi(1-s)$ *for all* $\lambda < \mathrm{Re}(s) < \beta$,
(2) $\Psi(s) = O(e^{(\frac{\pi}{4} - \omega + \epsilon)|t|})$ *for every positive ϵ and uniformly for* $\lambda < \mathrm{Re}(s) < \beta$.

Lemma 2.5. *Let $\phi, \psi \in K(\omega, \alpha)$ and Z be defined by* (1.8)*. Then we have*

$$Z(1-s) = Z(s).$$

Proof. We know ϕ and ψ are cosine reciprocal which is a special case of functions reciprocal in the Hankel transform when $\nu = -1/2$. Then by Lemma 2.4 we have

$$Z_2(s) = Z_1(1-s) \quad \text{and} \quad Z_1(s) = Z_2(1-s),$$

where Z_1 and Z_2 are defined by (1.6). Finally, note that

$$Z(s) = Z_1(s) + Z_2(s) = Z_1(1-s) + Z_2(1-s) = Z(1-s), \tag{2.4}$$

as claimed. ∎

We will also use Stirling's formula for $\Gamma(s)$, $s = \sigma + it$, in a vertical strip $c \le \sigma \le d$ given by

$$|\Gamma(\sigma + it)| = (2\pi)^{\frac{1}{2}} |t|^{\sigma - \frac{1}{2}} e^{-\frac{1}{2}\pi |t|} \left(1 + O\left(\frac{1}{|t|}\right)\right) \tag{2.5}$$

as $|t| \to \infty$.

3. Zeros of $F(s)$: Proof of Theorem 1.1

Let λ_j be any real number. Replacing t by $t + \lambda_j$ in Lemma 2.1 we find

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\alpha t} \rho(t + \lambda_j)\, dt &= e^{-\alpha\lambda_j} \left(- 4\pi \cos\frac{\alpha}{2} + 2\pi e^{\frac{\alpha i}{2}} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi e^{2\alpha i}} \right) \\ &= -2\pi \left(e^{\frac{\alpha i}{2} - \alpha\lambda_j} + e^{-\frac{\alpha i}{2} - \alpha\lambda_j} - e^{\frac{\alpha i}{2} - \alpha\lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}} \right). \end{aligned} \tag{3.1}$$

Differentiating both sides of (3.1) $2m$ times with respect to α we get

$$\begin{aligned} \int_{-\infty}^{\infty} t^{2m} e^{\alpha t} \rho(t + \lambda_j)\, dt = -2\pi \Bigg(\left(\frac{i}{2} - \lambda_j\right)^{2m} e^{\frac{\alpha i}{2} - \alpha\lambda_j} + \left(\frac{i}{2} + \lambda_j\right)^{2m} e^{-\frac{\alpha i}{2} - \alpha\lambda_j} \\ - \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(e^{\frac{\alpha i}{2} - \alpha\lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}} \right) \Bigg). \end{aligned} \tag{3.2}$$

Let $\frac{i}{2} - \lambda_j = r_j e^{i\theta_j}$. Without loss of generality, one may take $0 < \theta_j < \frac{\pi}{2}$. From (3.2) we have

$$\begin{aligned} \int_{-\infty}^{\infty} t^{2m} e^{\alpha t} \rho(t + \lambda_j)\, dt &= -2\pi e^{-\alpha\lambda_j} \left(r_j^{2m} e^{i(\frac{\alpha}{2} + 2m\theta_j)} + r_j^{2m} e^{i(\frac{-\alpha}{2} + 2\pi m - 2m\theta_j)} \right) \\ &\quad + 2\pi \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(e^{\frac{\alpha i}{2} - \alpha\lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}} \right) \\ &= -4\pi e^{-\alpha\lambda_j} r_j^{2m} \cos\left(\frac{\alpha}{2} + 2m\theta_j\right) \\ &\quad + 2\pi \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(e^{\frac{\alpha i}{2} - \alpha\lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}} \right). \end{aligned} \tag{3.3}$$

Multiplying both sides of (3.3) by c_j and summing over j, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} t^{2m} e^{\alpha t} F\left(\frac{1}{2}+it\right) dt &= -4\pi \sum_{j=1}^{\infty} c_j e^{-\alpha\lambda_j} r_j^{2m} \cos\left(\frac{\alpha}{2}+2m\theta_j\right) \\ &\quad + 2\pi \sum_{j=1}^{\infty} \frac{\partial^{2m}}{\partial\alpha^{2m}}\left(c_j e^{\frac{\alpha i}{2}-\alpha\lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}}\right) \\ &=: -4\pi \sum_{j=1}^{\infty} h_j(\alpha) + 2\pi \sum_{j=1}^{\infty} \tilde{h}_j(\alpha). \end{aligned} \tag{3.4}$$

By Stirling's formula (2.5), we have

$$\rho(t) \ll |t|^A e^{-\frac{\pi}{4}|t|}$$

as $t \to \infty$, where A is a positive number. Since $\{\lambda_j\}$ is a bounded sequence, we find that

$$\sum_{j=1}^{\infty} c_j \rho(t+\lambda_j) \ll |t|^A e^{-\frac{\pi}{4}|t|} \sum_{j=1}^{\infty} |c_j| \ll |t|^A e^{-\frac{\pi}{4}|t|} \tag{3.5}$$

as $t \to \infty$. Hence (3.5) justifies the interchange of summation and integration on the left-hand side of (3.4) for $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$. Also for the same bounded sequence $\{\lambda_j\}$, and for any given bounded interval I_0 with $\alpha \in I_0$, we have

$$\sum_{j=1}^{\infty} ||h_j(\alpha)||_\infty \leq \sum_{j=1}^{\infty} |c_j| \max_{\alpha,j}\{r_j^{2m} e^{-\alpha\lambda_j}\} \ll 1 \tag{3.6}$$

uniformly for $\alpha \in I_0$.

Let m be a fixed non-negative integer. Let $\epsilon > 0$ be any number such that $-\frac{\pi}{4}+\epsilon \leq \alpha \leq \frac{\pi}{4}-\epsilon$. For any $0 \leq l \leq m$, we observe that

$$\sum_{n=-\infty}^{\infty} n^{2l} e^{-n^2\pi\cos(2\alpha)} \ll_\epsilon 1, \tag{3.7}$$

and for any bounded sequence $\{\lambda_j\}$,

$$\frac{\partial^l}{\partial\alpha^l} e^{\frac{\alpha i}{2}-\alpha\lambda_j} \ll_{\epsilon,m} 1. \tag{3.8}$$

Using Leibniz's rule, (3.7) and (3.8), we have $|\tilde{h}_j(\alpha)| \ll_{\epsilon,m} |c_j|$. Hence

$$\sum_{j=1}^{\infty} ||\tilde{h}_j(\alpha)||_\infty \ll_{\epsilon,m} 1 \tag{3.9}$$

for $-\frac{\pi}{4}+\epsilon \leq \alpha \leq \frac{\pi}{4}-\epsilon$. Let

$$h(\alpha) := \sum_{j=1}^{\infty} c_j e^{-\alpha\lambda_j}.$$

Note that $h(\alpha)$ is an entire function. From (3.9), we find that

$$\sum_{j=1}^{\infty} \frac{\partial^{2m}}{\partial\alpha^{2m}}\left(c_j e^{\frac{\alpha i}{2}-\alpha\lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}}\right) = \frac{\partial^{2m}}{\partial\alpha^{2m}}\left(h(\alpha) e^{\frac{\alpha i}{2}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}}\right)$$

for $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$. Therefore by Lemma 2.2, we deduce that

$$\lim_{\alpha \to \frac{\pi}{4}^-} \sum_{j=1}^{\infty} \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(c_j e^{\frac{\alpha i}{2} - \alpha \lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi e^{2\alpha i}} \right) = 0. \tag{3.10}$$

Letting $\alpha \to \frac{\pi}{4}^-$ on both sides of (3.4), and using (3.6) and (3.10) we get

$$\lim_{\alpha \to \frac{\pi}{4}^-} \int_{-\infty}^{\infty} t^{2m} e^{\alpha t} F\left(\frac{1}{2} + it\right) dt = -4\pi \sum_{j=1}^{\infty} c_j e^{\frac{-\pi \lambda_j}{4}} r_j^{2m} \cos\left(\frac{\pi}{8} + 2m\theta_j\right). \tag{3.11}$$

By hypothesis, there exists a positive integer M such that

$$|\lambda_M| = \max_j \{|\lambda_j|\} \quad \text{and} \quad \lambda_M \neq \lambda_j \quad \text{for} \quad M \neq j.$$

Then the right-hand side of (3.11) can be written as

$$-4\pi c_M r_M^{2m} e^{-\frac{\pi \lambda_M}{4}} \cos\left(\frac{\pi}{8} + 2m\theta_M\right)(1 + E(X) + H(X)), \tag{3.12}$$

where

$$E(X) := \sum_{\substack{j \neq M \\ j \leq X}} \frac{c_j}{c_M} e^{-\frac{\pi}{4}(\lambda_j - \lambda_M)} \left(\frac{r_j}{r_M}\right)^{2m} \frac{\cos(\frac{\pi}{8} + 2m\theta_j)}{\cos(\frac{\pi}{8} + 2m\theta_M)}, \tag{3.13}$$

as well as

$$H(X) := \sum_{\substack{j \neq M \\ j > X}} \frac{c_j}{c_M} e^{-\frac{\pi}{4}(\lambda_j - \lambda_M)} \left(\frac{r_j}{r_M}\right)^{2m} \frac{\cos(\frac{\pi}{8} + 2m\theta_j)}{\cos(\frac{\pi}{8} + 2m\theta_M)}. \tag{3.14}$$

Next we claim that there exists a sequence such that for each value m in it, the inequality $|\cos(\frac{\pi}{8} + 2m\theta_M)| \geq \frac{1}{3}$ holds. Let $\frac{i}{2} - \lambda_M = r_M e^{i\theta_M}$ for $0 < \theta_M < \frac{\pi}{2}$. Then

$$r_M > r_j \quad \text{for} \quad M \neq j. \tag{3.15}$$

Now we divide the proof of the claim into two cases. First consider the case when $\frac{\theta_M}{\pi}$ is irrational. Then by Lemma 2.3, we find two subsequences $\{p_n\}$ and $\{q_n\}$ such that

$$\left(\frac{p_n \theta_M}{\pi}\right) \to \frac{1}{2^4} \quad \text{and} \quad \left(\frac{q_n \theta_M}{\pi}\right) \to \frac{1}{2}$$

for $n \to \infty$, where, as before, (x) denotes the fractional part of x. One can see that for $n \to \infty$,

$$\cos\left(\frac{\pi}{8} + 2p_n\theta_M\right) \to \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos\left(\frac{\pi}{8} + 2q_n\theta_M\right) \to -\cos\left(\frac{\pi}{8}\right) < -\frac{1}{3}. \tag{3.16}$$

In the second case, we consider $\frac{\theta_M}{\pi} := \frac{p}{q}$ to be rational. Since $0 < \frac{p}{q} < \frac{1}{2}$, there exists an integer n_0 such that $\frac{1}{4} \leq n_0 \lfloor \frac{q}{2p} \rfloor \frac{p}{q} < \frac{1}{2}$. Now define $p_n := nq$ and $q_n := nq + n_0 \lfloor \frac{q}{2p} \rfloor$. Therefore for all n,

$$\cos\left(\frac{\pi}{8} + p_n\theta_M\right) = \cos\left(\frac{\pi}{8}\right) \quad \text{and} \quad -\cos\left(\frac{\pi}{8}\right) < \cos\left(\frac{\pi}{8} + q_n\theta_M\right) \leq \cos\left(\frac{5\pi}{8}\right). \tag{3.17}$$

The above constructions show that if m runs through the sequence $\{p_n\} \cup \{q_n\}$, then for large m, we have

$$\left| \cos\left(\frac{\pi}{8} + 2m\theta_M\right) \right| \geq \frac{1}{3}. \tag{3.18}$$

Let m be any large integer from the sequence $\{p_n\} \cup \{q_n\}$. From (3.14), (3.15) and (3.18) we have

$$H(X) \leq \frac{3}{|c_M|} \sum_{\substack{j \neq M \\ j > X}} |c_j| < \frac{1}{1914}, \tag{3.19}$$

for a large X. Let

$$c_X = \max_{j \leq X} \left\{ \frac{|r_j|}{|r_M|} \right\}.$$

Since X is finite, by (3.15) we find that $c_X < 1$. Similarly for large $m \in \{p_n\} \cup \{q_n\}$, using (3.13) and (3.18) we have

$$E(X) \leq 3 \frac{c_X^{2m}}{|c_M|} \sum_{\substack{j \neq M \\ j \leq X}} |c_j|, \tag{3.20}$$

which tends to 0 as $m \to \infty$ through the sequence $\{p_n\} \cup \{q_n\}$. Also by construction $\cos(\frac{\pi}{8} + m\theta_M)$ changes sign infinitely often for infinitely many values of $m \in \{p_n\} \cup \{q_n\}$. Hence from (3.12), (3.19), and (3.20), we see that the right-hand side of (3.11) changes sign infinitely often for infinitely many values of $m \in \{p_n\} \cup \{q_n\}$.

Let us now suppose that $F(s)$ has only finitely many zeros on the line $\sigma = \frac{1}{2}$, and hence that $F(\frac{1}{2} + it)$ never changes sign for $|t| > T$ for some large T. In other words, we can say that $F(\frac{1}{2} + it) > 0$ for $|t| > T$, or that $F(\frac{1}{2} + it) < 0$ for $|t| > T$, or that $F(\frac{1}{2} + it)$ takes opposite signs in $t > T$ and $t < T$. First of all, let us consider that $F(\frac{1}{2} + it) > 0$ for $|t| > T$.

Next, let us define the quantity L by the equation

$$L := \lim_{\alpha \to \frac{\pi}{4}^-} \int_{|t| \geq T} F\left(\frac{1}{2} + it\right) t^{2m} e^{\alpha t}\, dt. \tag{3.21}$$

Since the integrand in the above integral is positive, one sees that for any $T' > T$,

$$\lim_{\alpha \to \frac{\pi}{4}} \int_{T \leq |t| \leq T'} F\left(\frac{1}{2} + it\right) t^{2m} e^{\alpha t}\, dt \leq \lim_{\alpha \to \frac{\pi}{4}} \int_{|t| > T} F\left(\frac{1}{2} + it\right) t^{2m} e^{\alpha t}\, dt = L. \tag{3.22}$$

In particular,

$$\int_{T \leq |t| \leq T'} F\left(\frac{1}{2} + it\right) t^{2m} e^{\frac{\pi}{4} t}\, dt \leq L. \tag{3.23}$$

Hence

$$\int_{-\infty}^{\infty} F\left(\frac{1}{2} + it\right) t^{2m} e^{\frac{\pi}{4} t}\, dt \tag{3.24}$$

is convergent.

Thus, for every $m \in \mathbb{N}$ we have

$$\int_{-\infty}^{\infty} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t}\, dt = -4\pi \sum_{j=1}^{\infty} c_j e^{\frac{-\pi\lambda_j}{4}} r_j^{2m} \cos\left(\frac{\pi}{8}+2m\theta_j\right). \tag{3.25}$$

This is impossible since the right-hand side switches sign infinitely often. We can find an integer $m \in \{p_n\} \cup \{q_n\}$ such that

$$\begin{aligned} \int_{|t|\geq T} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t}\, dt &< -\int_{-T}^{T} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t}\, dt \\ &< T^{2m} \int_{-T}^{T} \left| F\left(\frac{1}{2}+it\right) e^{\frac{\pi}{4}t} \right| dt \\ &\leq T^{2m} R. \end{aligned} \tag{3.26}$$

It is seen that R is independent of m.

Finally, by the hypothesis on $F(\frac{1}{2}+it)$, we see that there exists $\varepsilon = \varepsilon(T) > 0$ such that $F(\frac{1}{2}+it) \geq \varepsilon$ for all $2T < t < 2T+1$. Hence

$$\begin{aligned} \int_{|t|\geq T} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t}\, dt &\geqslant \int_{2T}^{2T+1} \varepsilon t^{2m} e^{\frac{\pi}{4}t}\, dt \\ &\geqslant \int_{2T}^{2T+1} \varepsilon t^{2m}\, dt \\ &= \varepsilon\left(\frac{(2T+1)^{2m+1}}{2m+1} - \frac{(2T)^{2m+1}}{2m+1} \right) \\ &\geqslant \varepsilon (2T)^{2m}. \end{aligned} \tag{3.27}$$

Combining these two results, we have

$$\varepsilon(2T)^{2m} \leqslant \int_{|t|\geq T} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t}\, dt < -\int_{-T}^{T} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t}\, dt < T^{2m} R, \tag{3.28}$$

for infinitely values of $m \in \{p_n\} \cup \{q_n\}$. This is equivalent to

$$2^{2m} < \frac{R}{\varepsilon}$$

holding for infinitely many values of $m \in \{p_n\} \cup \{q_n\}$. However this is impossible since m can be taken to be arbitrarily large.

Now, if $F(\frac{1}{2}+it) < 0$ for $|t| > T$, we multiply both sides of (3.11) by -1 and carry out similar arguments for $-F(\frac{1}{2}+it)$. Lastly, if $F(\frac{1}{2}+it)$ takes opposite signs in $t > T$ and $t < -T$ then we differentiate (3.1) $2m+1$ times with respect to α, instead of $2m$ times. In this case, (3.11) and (3.15) can be proved similarly. If $\frac{\theta_M}{\pi}$ is irrational, then we construct the sequences $\{p_n\}$ and $\{q_n\}$ such that

$$\left(\frac{p_n\theta_M}{\pi}\right) \to \frac{1}{2^4} - \frac{\theta_M}{2\pi} \quad \text{and} \quad \left(\frac{q_n\theta_M}{\pi}\right) \to \frac{1}{2} - \frac{\theta_M}{2\pi}.$$

If $\frac{\theta_M}{\pi}$ is rational, then we construct the sequences $\{p_n\}$ and $\{q_n\}$ by

$$p_n := nq - \frac{1}{2} \quad \text{and} \quad q_n := nq + n_0 \left\lfloor \frac{q}{2p} \right\rfloor - \frac{1}{2}.$$

For the above $\{p_n\} \cup \{q_n\}$ one can check easily that (3.16), (3.17) and (3.18) hold when we replace $2m$ by $2m+1$. Likewise, one can use similar arguments to prove (3.28) for $2m+1$ and arrive at a contradiction. Hence we have proved the theorem.

4. The theta transformation formula and proof of Theorem 1.2

Let $\phi, \psi \in K(\omega, \alpha)$ for $\omega > 0$. Suppose that Θ and Z are defined as in (1.8). Let us consider the following integral

$$\mathcal{H}(\Theta) := \int_0^\infty f(t)\Xi(t) Z\left(\frac{1}{2} + it\right) dt, \tag{4.1}$$

where

$$f(t) = g(it)g(-it) \tag{4.2}$$

with g an analytic function of t. By Stirling's formula (2.5), we have

$$\Xi(t) \ll t^A e^{-\frac{\pi}{4}t}$$

for a positive constant A. By Lemma 2.4 and $\nu = -1/2$ we have

$$Z(\sigma + it) \ll e^{(\frac{\pi}{4} - \omega + \epsilon)t}$$

for every $\epsilon > 0$. Therefore the integral in (4.1) will be convergent as long as $f(t) \ll t^B$ for some positive constant B. From Lemma 2.5, we observe $Z(\frac{1}{2} + it)$ is an even function of t. Since $\Xi(t)$ is an even function, (4.1) can be written as

$$\begin{aligned} \mathcal{H}(\Theta) &= \frac{1}{2} \int_{-\infty}^\infty g(it)g(-it)\Xi(t) Z\left(\frac{1}{2} + it\right) dt \\ &= \frac{1}{2i} \int_{(\frac{1}{2})} g\left(s - \frac{1}{2}\right) g\left(\frac{1}{2} - s\right) \xi(s) Z(s) ds \end{aligned} \tag{4.3}$$

by the change $it = s - \frac{1}{2}$. Now let

$$g(s) = \frac{1}{s + \frac{1}{2}}, \quad \text{so that} \quad g\left(s - \frac{1}{2}\right) g\left(\frac{1}{2} - s\right) = \frac{1}{(1-s)s}.$$

This simplifies the expression for $\mathcal{H}$ in (4.3) to

$$\mathcal{H}(\Theta) = -\frac{1}{4i} \int_{(\frac{1}{2})} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) Z(s) ds.$$

The next step is to move the path of integration from the critical line $\mathrm{Re}(s) = \frac{1}{2}$ past the region of absolute convergence of $\zeta(s)$ at $s = 1 + \varepsilon$ with $\varepsilon > 0$. In doing so, we pick up a contribution of a possible simple pole at $s = 1$ so that

$$\begin{aligned} \mathcal{H}(\Theta) &= -\frac{1}{4i} \int_{(1+\varepsilon)} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) Z(s) ds + 2\pi i \frac{1}{4i} \operatorname*{res}_{s=1} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) Z(s) \\ &= -\frac{\pi}{2} \sum_{n=1}^\infty \frac{1}{2\pi i} \int_{(1+\varepsilon)} \Gamma\left(\frac{s}{2}\right) (\pi^{1/2} n)^{-s} Z(s) ds + \frac{\pi}{2} Z(1) \\ &= -\frac{\pi}{2} \sum_{n=1}^\infty \Theta(\pi^{1/2} n) + \frac{\pi}{2} Z(1). \end{aligned}$$

By Stirling's formula (2.5) and Lemma 2.4 with $\nu = -1/2$ we see that

$$\Gamma\left(\frac{s}{2}\right)Z(s) \ll e^{(-\omega+\epsilon)t} \tag{4.4}$$

for any $\epsilon > 0$. This is enough to justify the vanishing of the integrals along the horizontal segments of the rectangular contour as $t \to \infty$. We have thus shown that

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} Z\left(\frac{1}{2}+it\right)dt = \frac{\pi}{2}Z(1) - \frac{\pi}{2}\sum_{n=1}^\infty \Theta(\pi^{1/2}n).$$

We now give a special case of the above identity arising from a specific choice of the cosine reciprocal functions $\phi(x)$ and $\psi(x)$.

4.1. **Proof of Corollary 1.2.** Let $\theta, h \in \mathbb{C}$ be fixed parameters with $-\frac{\pi}{2} < \arg\theta < \frac{\pi}{2}$. If we set

$$\phi_h(x,\theta) = \exp(-\theta x^2)\cos(hx),$$

then its cosine reciprocal is then given by

$$\psi_h(x,\theta) = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp(-\theta u^2)\cos(hu)\cos(2ux)du = \theta^{-1/2}e^{-h^2/(4\theta)}\exp\left(-\frac{x^2}{\theta}\right)\cosh\left(\frac{hx}{\theta}\right).$$

It is clear that $\phi, \psi \in K(\omega, \alpha)$. The sum is given by

$$\begin{aligned}\Theta(x) = \Theta_h(x,\theta) &= \phi_h(x,\theta) + \psi_h(x,\theta) \\ &= \exp(-\theta x^2)\cos(hx) + \theta^{-1/2}e^{-h^2/(4\theta)}\exp\left(-\frac{x^2}{\theta}\right)\cosh\left(\frac{hx}{\theta}\right).\end{aligned}$$

Then

$$Z_1(s) = \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty x^{s-1}\phi_h(x,\theta)dx = \frac{1}{2}\theta^{-s/2}{}_1F_1\left(\frac{s}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right)$$

and

$$Z_2(s) = \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty x^{s-1}\psi_h(x,\theta)dx = \frac{1}{2}\theta^{(s-1)/2}{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right),$$

so that

$$\begin{aligned}Z(s) = Z_h(s,\theta) &= Z_1(s) + Z_2(s) \\ &= \frac{1}{2}\theta^{-s/2}{}_1F_1\left(\frac{s}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right) + \frac{1}{2}\theta^{(s-1)/2}{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right).\end{aligned}$$

Let us look at the infinite sum

$$\begin{aligned}\Delta(h,\theta) &= \sum_{n=1}^\infty \Theta_h(\pi^{1/2}n,\theta) \\ &= \sum_{n=1}^\infty \exp(-\pi\theta n^2)\cos(\pi^{1/2}hn) + \theta^{-1/2}e^{-h^2/(4\theta)}\sum_{n=1}^\infty \exp\left(-\frac{\pi n^2}{\theta}\right)\cosh\left(\frac{\pi^{1/2}hn}{\theta}\right).\end{aligned}$$

Define the first infinite sum in the last line by

$$\varpi(h,\theta) := \sum_{n=1}^\infty \exp(-\pi\theta n^2)\cos(\pi^{1/2}hn).$$

The functional equation of ϖ is given by [31, p. 124, Exercise 18]

$$2\varpi(h,\theta)+1=\theta^{-1/2}\exp\left(-\frac{h^2}{4\theta}\right)(2\varpi(ih\theta^{-1},\theta^{-1})+1).$$

Going back to Δ, we have

$$\begin{aligned}\Delta(h,\theta)&=\varpi(h,\theta)+\theta^{-1/2}e^{-h^2/(4\theta)}\varpi(ih\theta^{-1},\theta^{-1})\\&=2\varpi(h,\theta)+\frac{1}{2}-\frac{1}{2}\theta^{-1/2}e^{-h^2/(4\theta)}.\end{aligned}\tag{4.5}$$

We also note that

$$Z(1)=Z_h(1,\theta)=\frac{1}{2}+\frac{1}{2}\theta^{-1/2}e^{-h^2/(4\theta)}\tag{4.6}$$

and

$$\begin{aligned}Z\left(\frac{1}{2}+it\right)&=Z_h\left(\frac{1}{2}+it,\theta\right)\\&=\frac{1}{2}\left(\theta^{-\frac{1}{4}-\frac{it}{2}}{}_1F_1\left(\frac{1}{4}+\frac{it}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right)+\theta^{-\frac{1}{4}+\frac{it}{2}}{}_1F_1\left(\frac{1}{4}-\frac{it}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right)\right)\\&=\frac{1}{2}\theta^{-\frac{1}{4}}\nabla(h,t,\theta).\end{aligned}\tag{4.7}$$

Replacing (4.5),(4.6) and (4.7) in (1.11), and then letting $\theta=e^{-2x}$, we obtain (1.13). This proves Corollary 1.2.

4.2. **Proof of Corollary 1.1.** Just let $h=0$ in Corollary 1.2.

Remark 4.1. If we substitute $\nu=i\alpha$, with $\alpha\in\mathbb{R}$, in (1.2), write $\cos(i\alpha t)=\frac{1}{2}(e^{-\alpha t}+e^{\alpha t})$ to simplify the integral on the left, and use (1.4), one obtains Lemma 2.1.

Remark 4.2. The choice $(\phi(x),\psi(x))=\left(e^{-\theta x},\frac{2}{\sqrt{\pi}}\frac{\theta}{\theta^2+4x^2}\right)$, $-\frac{\pi}{2}<\arg\theta<\frac{\pi}{2}$, in (1.11) yields

$$\int_0^\infty\frac{\Xi(t)}{t^2+\frac{1}{4}}Z\left(\frac{1}{2}+it,\theta\right)\,dt=-\pi\left(\frac{1}{e^{\sqrt{\pi}\theta}-1}-\frac{1}{\sqrt{\pi}\theta}\right),\tag{4.8}$$

which is a rephrasing of the well-known identity [26, p. 23, Equation (2.7.1)], namely,

$$\int_0^\infty x^{s-1}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)dx=\Gamma(s)\zeta(s),\ \ (0<\mathrm{Re}(s)<1).\tag{4.9}$$

Equation (4.8) can be obtained from (4.9) using the functional equation for $\zeta(s)$.

5. Generalization of Hardy's formula: Proof of Theorem 1.3

Let $g(s)=\frac{1}{4\sqrt{2\pi}}\Gamma(\frac{1}{4}+\frac{s}{2})\Gamma(\frac{-1}{4}+\frac{s}{2})$ in (4.2) so that

$$f(t)=\frac{1}{32\pi^2}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}-\frac{it}{2}\right)=\frac{1}{(1+4t^2)\cosh\pi t}$$

and

$$g\left(s-\frac{1}{2}\right)g\left(\frac{1}{2}-s\right)=\frac{1}{4\pi}\Gamma(-s)\Gamma(s-1).\tag{5.1}$$

From (4.1) and (4.3), we have

$$\begin{aligned}\mathcal{H}(\Theta) &= \frac{1}{8\pi i}\int_{(\frac{1}{2})}\Gamma(-s)\Gamma(s-1)\frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)Z(s)\,ds \\ &= \frac{-1}{16i}\int_{(\frac{1}{2})}\frac{\zeta(s)}{\sin\pi s}\Gamma\left(\frac{s}{2}\right)Z(s)\pi^{-\frac{s}{2}}\,ds.\end{aligned} \tag{5.2}$$

Shifting the line of integration to Re $s = 1+\delta$ where $0 < \delta < 1$, and considering the contribution of the pole of order 2 at $s = 1$, we have

(5.3)

$$\begin{aligned}\mathcal{H}(\Theta) &= \int_0^\infty \frac{\Xi(t)}{(1+4t^2)}\frac{Z(\frac{1}{2}+it)}{\cosh\pi t}\,dt = \frac{1}{2}\int_0^\infty \frac{\Xi(\frac{t}{2})}{(1+t^2)}\frac{Z(\frac{1+it}{2})}{\cosh\frac{1}{2}\pi t}\,dt \\ &= \frac{-1}{16i}\left\{\sum_{n=1}^{\infty}\int_{(1+\delta)}\frac{Z(s)}{\sin\pi s}\Gamma\left(\frac{s}{2}\right)(\sqrt{\pi}n)^{-s}\,ds - 2\pi i\left(-\frac{Z'(1)}{\pi} - \frac{(\gamma-\log 4\pi)}{2\pi}Z(1)\right)\right\}.\end{aligned}$$

Note that

$$|\sin(\pi(\sigma+it))| \geq \frac{e^{\pi|t|}}{2}\left(1-e^{-2\pi|t|}\right) > \frac{e^{\pi|t|}}{4}$$

for large t. Then from (4.4), we find that the integrals along the horizontal segments of the rectangular contour go to 0 as $t\to\infty$. It is well-known [19, p. 91, Equation (3.3.10)] that for $0 < \text{Re } s < 1$,

$$\frac{1}{2\pi i}\int_{(c)}\frac{x^{-s}}{\sin\pi s}\,ds = \frac{1}{\pi(1+x)}. \tag{5.4}$$

Also, from [19, p. 83, Equation (3.1.13)], we have

$$\frac{1}{2\pi i}\int_{(c)}F(s)G(s)w^{-s}\,ds = \int_0^\infty f(x)g\left(\frac{w}{x}\right)\frac{dx}{x}, \tag{5.5}$$

where $F(s)$ and $G(s)$ are Mellin transforms of $f(x)$ and $g(x)$ respectively. Hence, shifting the line of integration to Re $s = c, 0 < c < 1$, using (1.9) and the fact that $\phi,\psi\in K(\omega,0,\beta)$, we find that

(5.6)

$$\int_0^\infty \frac{\Xi(\frac{t}{2})}{(1+t^2)}\frac{Z(\frac{1+it}{2})}{\cosh\frac{1}{2}\pi t}\,dt = \frac{-1}{4}\left\{\sum_{n=1}^{\infty}\left(\int_0^\infty\frac{\Theta(x)}{x+n\sqrt{\pi}}\,dx - \frac{Z(1)}{n}\right) + Z'(1) + \frac{(\gamma-\log 4\pi)}{2}Z(1)\right\},$$

which completes the proof of Theorem 1.3.

We now prove Hardy's formula as a special case of the above theorem.

Proof of Corollary 1.3. Let $\phi(x) = e^{-a^2x^2}$ in the first equation in (1.5). It is easy to see that $\psi(x) = \frac{1}{a}e^{-x^2/a^2}$, and that $\phi(x)$ and $\psi(x)$ are reciprocal in the Fourier cosine transform. This gives

$$\Theta(x) = e^{-a^2x^2} + \frac{1}{a}e^{-x^2/a^2}.$$

Also one can check that

$$Z(1) = \frac{1}{2}\left(1+\frac{1}{a}\right),\quad Z'(1) = \frac{\log a}{2}\left(1-\frac{1}{a}\right),\quad Z\left(\frac{1+it}{2}\right) = \frac{1}{\sqrt{a}}\cos\left(\frac{1}{2}t\log a\right). \tag{5.7}$$

Now

(5.8)
$$\begin{aligned}\int_0^\infty \frac{\Theta(x)}{x+n\sqrt{\pi}}\,dx-\frac{Z(1)}{n}&=\left(\int_0^\infty \frac{e^{-a^2x^2}}{x+n\sqrt{\pi}}\,dx-\frac{1}{2na}\right)+\frac{1}{a}\left(\int_0^\infty \frac{e^{-x^2/a^2}}{x+n\sqrt{\pi}}\,dx-\frac{a}{2n}\right)\\&=-\int_0^\infty e^{-\pi a^2x^2}\left(\frac{1}{n}-\frac{1}{x+n}\right)dx-\frac{1}{a}\int_0^\infty e^{-\pi x^2/a^2}\left(\frac{1}{n}-\frac{1}{x+n}\right)dx,\end{aligned}$$

using the fact that $\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$. Hence,

$$\begin{aligned}&\sum_{n=1}^\infty\left(\int_0^\infty \frac{\Theta(x)}{x+n\sqrt{\pi}}\,dx-\frac{Z(1)}{n}\right)\\&=-\int_0^\infty e^{-\pi a^2x^2}(\psi(1+x)+\gamma)\,dx-\frac{1}{a}\int_0^\infty e^{-\pi x^2/a^2}(\psi(1+x)+\gamma)\,dx,\end{aligned}\tag{5.9}$$

by the interchange of the order of summation and integration, which is valid because of absolute convergence, and since [25, p. 54, Equation (3.10)]

$$\psi(x):=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma-\sum_{m=0}^\infty\left(\frac{1}{m+x}-\frac{1}{m+1}\right).\tag{5.10}$$

Now we use the integral evaluation

$$\int_0^\infty e^{-\pi a^2x^2}(\gamma+\log x)\,dx=\frac{\gamma-\log(4\pi a^2)}{4a}\tag{5.11}$$

in (5.9) to write

$$\begin{aligned}&\sum_{n=1}^\infty\left(\int_0^\infty \frac{\Theta(x)}{x+n\sqrt{\pi}}\,dx-\frac{Z(1)}{n}\right)\\&=-\frac{\gamma-\log(4\pi a^2)}{4a}-\frac{(\gamma-\log(4\pi/a^2))}{4}\\&\quad-\int_0^\infty e^{-\pi a^2x^2}(\psi(1+x)-\log x)\,dx-\frac{1}{a}\int_0^\infty e^{-\pi x^2/a^2}(\psi(1+x)-\log x)\,dx.\end{aligned}\tag{5.12}$$

Next we show that

$$\int_0^\infty e^{-\pi x^2/a^2}(\psi(1+x)-\log x)\,dx=a\int_0^\infty e^{-\pi a^2x^2}(\psi(1+x)-\log x)\,dx.\tag{5.13}$$

To that end, note that

$$e^{-\pi x^2/a^2}=2a\int_0^\infty e^{-\pi a^2y^2}\cos(2\pi yx)\,dy.\tag{5.14}$$

Hence,

(5.15)
$$\begin{aligned}\int_0^\infty e^{-\pi x^2/a^2}(\psi(1+x)-\log x)\,dx&=2a\int_0^\infty\int_0^\infty e^{-\pi a^2y^2}\cos(2\pi yx)(\psi(1+x)-\log x)\,dy\,dx\\&=2a\int_0^\infty e^{-\pi a^2y^2}\,dy\int_0^\infty(\psi(1+x)-\log x)\cos(2\pi yx)\,dx,\end{aligned}$$

where the interchange of the order of integration can again be justified. From page 220 in Ramanujan's Lost Notebook [22], we see that the function $\psi(1+x)-\log x$ is reciprocal (up to a constant) in the Fourier-cosine transform, namely,

$$\int_0^\infty (\psi(1+x)-\log x)\cos(2\pi yx)\,dx = \frac{1}{2}(\psi(1+y)-\log y). \tag{5.16}$$

This property was later rediscovered by Guinand [9] in 1947.

Substituting (5.16) in (5.15), we obtain (5.13). Now substitute (5.13) in (5.12) to obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\int_0^\infty \frac{\Theta(x)}{x+n\sqrt{\pi}}\,dx - \frac{Z(1)}{n}\right)\\ = -\frac{\gamma-\log(4\pi a^2)}{4a} - \frac{(\gamma-\log(4\pi/a^2))}{4} - 2\int_0^\infty e^{-\pi a^2x^2}(\psi(1+x)-\log x)\,dx.\end{aligned} \tag{5.17}$$

Finally from (5.6), (5.7) and (5.17), we obtain Hardy's formula (1.15). □

Remark 5.1. Theorem 1.3 in [5] can also be obtained as a special case of Theorem 1.3 of this paper combining the methods in the proofs of Corollaries 1.2 and 1.3.

6. Generalization of Ferrar's formula: Proof of Theorem 1.4

Let $g(s) = \frac{\sqrt{2}}{\frac{1}{2}-s}\Gamma(\frac{1}{4}+\frac{s}{2})$ in (4.2) so that

$$f\left(\frac{t}{2}\right) = g\left(\frac{it}{2}\right)g\left(\frac{-it}{2}\right) = \frac{8}{1+t^2}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right).$$

Thus from (4.1) and (4.3),

$$\begin{aligned}\mathcal{H}(\Theta) &= 4\int_0^\infty \Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\frac{\Xi(t/2)}{1+t^2}Z\left(\frac{1+it}{2}\right)dt\\ &= \frac{1}{2i}\int_{(\frac{1}{2})}\frac{2}{(1-s)s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\xi(s)Z(s)\,ds\\ &= -\frac{1}{2i}\int_{(\frac{1}{2})}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)Z(s)\pi^{-s/2}\,ds.\end{aligned} \tag{6.1}$$

We now apply the residue theorem after shifting the line of integration to $\mathrm{Re}\, s = 1+\delta$ where $0<\delta<2$ and considering the contribution of the pole of order 2 at $s=1$. Using (2.5) for the gamma functions $\Gamma(s/2)$ and $\Gamma(1-s/2)$ and (4.4), it is seen that the integrals along the horizontal segments go to zero as $t\to\infty$. This gives

$$\begin{aligned}\mathcal{H}(\Theta) = -\frac{1}{2i}\Bigg\{&\sum_{n=1}^{\infty}\int_{(1+\delta)}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)Z(s)(\sqrt{\pi}n)^{-s}\,ds\\ &- 2\pi i\lim_{s\to1}\frac{d}{ds}\left((s-1)^2\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)Z(s)\pi^{-s/2}\right)\Bigg\}\\ = -\frac{1}{2i}\Bigg\{&-2\pi i\left(-2\sqrt{\pi}Z'(1)-\sqrt{\pi}(\gamma-\log 16\pi)Z(1)\right)\\ &+\sqrt{\pi}\sum_{n=1}^{\infty}\left(\int_{(c)}B\left(\frac{s}{2},\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)Z(s)(\sqrt{\pi}n)^{-s}\,ds\right.\end{aligned} \tag{6.2}$$

$$+\frac{2\pi i}{\sqrt{\pi}}\lim_{s\to 1}(s-1)\Gamma\left(\frac{1-s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)Z(s)(\sqrt{\pi}n)^{-s}\Big)\Big\},$$

where $B(s,z-s)$ is the Euler beta integral given by

$$B(s,z-s)=\int_0^\infty \frac{x^{s-1}}{(1+x)^z}\,dx=\frac{\Gamma(s)\Gamma(z-s)}{\Gamma(z)},\quad 0<\text{ Re }s<\text{ Re }z. \tag{6.3}$$

From (6.3), we have for $0<c=\text{Re }s<1$,

$$\frac{1}{2\pi i}\int_{(c)}B\left(\frac{s}{2},\frac{1-s}{2}\right)x^{-s}\,ds=\frac{2}{\sqrt{1+x^2}}. \tag{6.4}$$

Along with (1.9), (5.5), and the fact that $\phi,\psi\in K(\omega,0,\beta)$, this gives

$$\begin{aligned}\mathcal{H}(\Theta)&=4\int_0^\infty \Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\frac{\Xi(t/2)}{1+t^2}Z\left(\frac{1}{2}+it\right)dt\\&=-\pi^{3/2}\left\{2Z'(1)+(\gamma-\log 16\pi)Z(1)+2\sum_{n=1}^\infty\left(\int_0^\infty\frac{\Theta(x)}{\sqrt{x^2+\pi n^2}}\,dx-\frac{Z(1)}{n}\right)\right\}.\end{aligned} \tag{6.5}$$

Now we prove Ferrar's formula as a special case of the above theorem.

Proof of Corollary 1.4. As in the proof of Corollary 1.3, we consider the pair of reciprocal functions $(\phi(x),\psi(x))=(e^{-a^2x^2},\frac{1}{a}e^{-x^2/a^2})$. From (5.7), we have

(6.6)

$$\begin{aligned}&\int_0^\infty\frac{\Theta(x)}{\sqrt{x^2+\pi n^2}}\,dx-\frac{Z(1)}{n}=\left(\int_0^\infty\frac{e^{-a^2x^2}}{\sqrt{x^2+\pi n^2}}\,dx-\frac{1}{2na}\right)+\frac{1}{a}\left(\int_0^\infty\frac{e^{-x^2/a^2}}{\sqrt{x^2+\pi n^2}}\,dx-\frac{a}{2n}\right)\\&=\int_0^\infty e^{-\frac{a^2t^2}{4\pi}}\left(\frac{1}{\sqrt{t^2+4\pi^2n^2}}-\frac{1}{2\pi n}\right)dt+\frac{1}{a}\int_0^\infty e^{-\frac{t^2}{4\pi a^2}}\left(\frac{1}{\sqrt{t^2+4\pi^2n^2}}-\frac{1}{2\pi n}\right)dt\end{aligned}$$

so that

(6.7)

$$\begin{aligned}&\sum_{n=1}^\infty\left(\int_0^\infty\frac{\Theta(x)}{\sqrt{x^2+\pi n^2}}\,dx-\frac{Z(1)}{n}\right)\\&=\int_0^\infty e^{-\frac{a^2t^2}{4\pi}}\sum_{n=1}^\infty\left(\frac{1}{\sqrt{t^2+4\pi^2n^2}}-\frac{1}{2\pi n}\right)dt+\frac{1}{a}\int_0^\infty e^{-\frac{t^2}{4\pi a^2}}\sum_{n=1}^\infty\left(\frac{1}{\sqrt{t^2+4\pi^2n^2}}-\frac{1}{2\pi n}\right)dt\\&=:J(a)+\frac{1}{a}J\left(\frac{1}{a}\right),\end{aligned}$$

say. Here the interchange of the order of summation and integration is justified by absolute convergence. From [30, Equation 6], we have, for Re $t>0$,

$$2\sum_{n=1}^\infty K_0(nt)=\pi\left\{\frac{1}{t}+2\sum_{n=1}^\infty\left(\frac{1}{\sqrt{t^2+4\pi^2n^2}}-\frac{1}{2n\pi}\right)\right\}+\gamma+\log\left(\frac{t}{2}\right)-\log 2\pi. \tag{6.8}$$

Hence,

(6.9)

$$J(a)=\int_0^\infty e^{-\frac{a^2t^2}{4\pi}}\left(\frac{1}{2\pi}\left(-\gamma+\log 4\pi-\log t+2\sum_{n=1}^\infty K_0(nt)\right)-\frac{1}{2t}\right)dt$$

$$= \frac{(-\gamma + \log 4\pi)}{2\pi}\frac{\pi}{a} - \frac{1}{2\pi}\bigg(- \frac{\pi}{2a}(\gamma - \log \pi + 2\log a)\bigg) + \frac{1}{\pi}\int_0^\infty e^{-\frac{a^2t^2}{4\pi}}\bigg(\sum_{n=1}^\infty K_0(nt) - \frac{\pi}{2t}\bigg)\,dt$$

$$= -\frac{\gamma}{4a} + \frac{\log(4\sqrt{\pi}a)}{2a} + \frac{1}{\pi}\int_0^\infty e^{-\frac{a^2t^2}{4\pi}}\bigg(\sum_{n=1}^\infty K_0(nt) - \frac{\pi}{2t}\bigg)\,dt.$$

Thus

$$J\bigg(\frac{1}{a}\bigg) = -\frac{\gamma a}{4} + \frac{a\log(\frac{4\sqrt{\pi}}{a})}{2} + \frac{1}{\pi}\int_0^\infty e^{-\frac{t^2}{4\pi a^2}}\bigg(\sum_{n=1}^\infty K_0(nt) - \frac{\pi}{2t}\bigg)\,dt. \tag{6.10}$$

Next we prove that

$$\int_0^\infty e^{-\frac{t^2}{4\pi a^2}}\bigg(\sum_{n=1}^\infty K_0(nt) - \frac{\pi}{2t}\bigg)\,dt = a\int_0^\infty e^{-\frac{a^2t^2}{4\pi}}\bigg(\sum_{n=1}^\infty K_0(nt) - \frac{\pi}{2t}\bigg)\,dt. \tag{6.11}$$

Using (5.14), we have

(6.12)

$$\begin{aligned}\int_0^\infty e^{-\frac{t^2}{4\pi a^2}}\bigg(\sum_{n=1}^\infty K_0(nt) - \frac{\pi}{2t}\bigg)\,dt &= \int_0^\infty \bigg(2a\int_0^\infty e^{-\pi a^2y^2}\cos(yt)\,dy\bigg)\bigg(\sum_{n=1}^\infty K_0(nt) - \frac{\pi}{2t}\bigg)\,dt\\ &= 2a\int_0^\infty e^{-\pi a^2y^2}\,dy\int_0^\infty \bigg(\sum_{n=1}^\infty K_0(nt) - \frac{\pi}{2t}\bigg)\cos(yt)\,dt.\end{aligned}$$

Now letting $x = t/(2\pi)$ in Lemma 1.5 below, using the resulting identity in the above equation and then again employing a change of variable $y = t/(2\pi)$, we obtain (6.11). Thus from (6.7), (6.9), (6.10) and (6.11), we deduce that

$$\begin{aligned}&\sum_{n=1}^\infty \bigg(\int_0^\infty \frac{\Theta(x)}{\sqrt{x^2+\pi n^2}}\,dx - \frac{Z(1)}{n}\bigg)\\ &= -\frac{\gamma}{4a} + \frac{\log(4\sqrt{\pi}a)}{2a} - \frac{\gamma}{4} + \frac{\log(\frac{4\sqrt{\pi}}{a})}{2} + \frac{2}{\pi}\int_0^\infty e^{-\frac{a^2t^2}{4\pi}}\bigg(\sum_{n=1}^\infty K_0(nt) - \frac{\pi}{2t}\bigg)\,dt.\end{aligned} \tag{6.13}$$

Finally, from (5.7), (6.5) and (6.13), we obtain Ferrar's formula (1.17). □

We conclude this paper with the proof of Lemma 1.5.

Proof of Lemma 1.5. We first show that the Mellin transform of $\sum_{n=1}^\infty K_0(2\pi nx) - \frac{1}{4x}$ for $0 < c = \text{Re } s < 1$ is given by

$$\int_0^\infty x^{s-1}\bigg(\sum_{n=1}^\infty K_0(2\pi nx) - \frac{1}{4x}\bigg)\,dx = \frac{1}{4\sqrt{\pi}}\Gamma\bigg(\frac{s}{2}\bigg)\Gamma\bigg(\frac{1-s}{2}\bigg)\zeta(1-s). \tag{6.14}$$

Observe that from (6.8), we have

$$\sum_{n=1}^\infty K_0(2\pi nx) - \frac{1}{4x} = \frac{\gamma}{2} + \frac{1}{2}\log\bigg(\frac{x}{2}\bigg) + \frac{1}{2}\sum_{n=1}^\infty\bigg(\frac{1}{\sqrt{x^2+n^2}} - \frac{1}{n}\bigg). \tag{6.15}$$

From (6.4), for $0 < c = \text{Re } s < 1$, we have

$$\frac{1}{2\pi i}\int_{(c)} \frac{1}{2\sqrt{\pi}}\Gamma\bigg(\frac{s}{2}\bigg)\Gamma\bigg(\frac{1-s}{2}\bigg)x^{-s}\,ds = \frac{1}{\sqrt{1+x^2}}. \tag{6.16}$$

Shifting the line of integration to $c' = \text{Re } s > 1$, we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{(c')}\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)x^{-s}\,ds &= \frac{1}{\sqrt{1+x^2}}+\lim_{s\to 1}\frac{(s-1)}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)x^{-s} \\ &= \frac{1}{\sqrt{1+x^2}}-\frac{1}{x}.\end{aligned}\tag{6.17}$$

Thus

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{1+(\frac{n}{x})^2}}-\frac{1}{n/x}\right) &= \sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_{(c')}\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{n}{x}\right)^{-s}ds \\ &= \frac{1}{2\pi i}\int_{(c')}\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)x^s\,ds.\end{aligned}\tag{6.18}$$

Shifting the line of integration back to $0 < c = \text{Re } s < 1$, we get

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{1+(\frac{n}{x})^2}}-\frac{1}{n/x}\right) &= \frac{1}{2\pi i}\int_{(c)}\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)x^s\,ds \\ &\quad+\lim_{s\to 1}\frac{d}{ds}\left(\frac{(s-1)^2}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)x^s\right) \\ &= \frac{1}{2\pi i}\int_{(c)}\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)x^s\,ds - x\left(\gamma+\log\left(\frac{x}{2}\right)\right),\end{aligned}\tag{6.19}$$

so that

$$\frac{\gamma}{2}+\frac{1}{2}\log\left(\frac{x}{2}\right)+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{x^2+n^2}}-\frac{1}{n}\right) = \frac{1}{2\pi i}\int_{(c)}\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)x^{s-1}\,ds.\tag{6.20}$$

Now replace s by $1-s$ in the above equation and then use (6.15) to complete the proof of (6.14). Now note that for $0 < c = \text{Re } s < 1$,

$$\int_0^{\infty} x^{s-1}\cos(2\pi yx)\,dx = (2\pi y)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s).\tag{6.21}$$

Now the Parseval formula [19, p. 83] in the theory of Mellin transforms says that if $F(s)$ and $G(s)$ are Mellin transforms of $f(x)$ and $g(x)$ respectively, and if the line $\text{Re } s = c$ lies in the common strip of analyticity of $F(1-s)$ and $G(s)$, then

$$\int_0^{\infty} f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{(c)} F(1-s)G(s)\,ds.\tag{6.22}$$

Now let $f(x) = \sum_{n=1}^{\infty} K_0(2\pi nx) - \frac{1}{4x}$, $g(x) = \cos(2\pi yx)$. Then from (6.14), (6.21) and (6.22), for $0 < c = \text{Re } s < 1$, we have

$$\begin{aligned}&\int_0^{\infty}\left(\sum_{n=1}^{\infty}K_0(2\pi nx)-\frac{1}{4x}\right)\cos(2\pi yx)\,dx \\ &= \frac{1}{2\pi i}\int_{(c)}\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)(2\pi y)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\,ds.\end{aligned}\tag{6.23}$$

Using the functional equation for $\zeta(s)$ in the form $\zeta(1-s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\cos(\frac{1}{2}\pi s)$ in the above equation and employing (6.14), we finally obtain (1.18). This completes the proof of the Lemma. □

Remark 6.1. Theorem 1.4 in [5] can also be obtained as a special case of Theorem 1.4 of this paper combining the methods in the proofs of corollaries 1.2 and 1.4.

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Department of Mathematics, Tulane University, New Orleans, LA 70118, USA
E-mail address: adixit@tulane.edu

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
E-mail address: nicolas.robles@math.uzh.ch

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: roy22@illinois.edu

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: zaharesc@illinois.edu