A TRANSFORMATION FORMULA INVOLVING THE GAMMA AND RIEMANN ZETA FUNCTIONS IN RAMANUJAN'S LOST NOTEBOOK

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In Memory of Alladi Ramakrishnan

1. Introduction

Pages 219–227 in the volume [15] containing Ramanujan's lost notebook are devoted to material "Copied from the Loose Papers." We emphasize that these pages are *not* part of the original lost notebook found by George Andrews at Trinity College Library, Cambridge in the spring of 1976. These "loose papers", in the handwriting of G.N. Watson, are found in the Oxford University Library, and evidently the original pages in Ramanujan's handwriting are no longer extant. Most of these nine pages, which are divided into four rough, partial manuscripts, are connected with material in Ramanujan's published papers. However, there is much that is new in these fragments, which will be completely examined in [2]. One claim in the first manuscript on pages 219–220 is the subject of this short note and is the most interesting theorem in the manuscript. This claim provides a beautiful series transformation involving the logarithmic derivative of the gamma function and the Riemann zeta function. To state Ramanujan's claim, it will be convenient to use the familiar notation [6, p. 952, formulas 8.360, 8.362, no. 1]

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{k+x} - \frac{1}{k+1}\right), \tag{1.1}$$

where γ denotes Euler's constant. We also need to recall the following functions associated with Riemann's zeta function $\zeta(s)$. Let

$$\xi(s) := (s-1)\pi^{-\frac{1}{2}s}\Gamma(1+\tfrac{1}{2}s)\zeta(s).$$

Then Riemann's Ξ-function is defined by

$$\Xi(\tfrac{1}{2}t) := \xi(\tfrac{1}{2} + \tfrac{1}{2}it).$$

Theorem 1.1. *Define*

$$\phi(x) := \psi(x) + \frac{1}{2x} - \log x. \tag{1.2}$$

If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\phi(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\phi(n\beta)\right\}$$
$$= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt, \tag{1.3}$$

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where γ denotes Euler's constant and $\Xi(x)$ denotes Riemann's Ξ-function.

The first identity in (1.3) is beautiful in its elegant symmetry and surprising as well, because why would subtracting the two leading terms in the asymptotic expansion of the logarithmic derivative of the Gamma function, in order to gain convergence of the infinite series on the left side, yield a "modular relation" for the resulting function? The second identity in (1.3) is also surprising, for why would the first identity foreshadow a connection with the Riemann zeta function in the second?

Although Ramanujan does not provide a proof of (1.3), he does indicate that (1.3) "can be deduced from"

$$\int_0^{\infty} (\psi(1+x) - \log x) \cos(2\pi n x) dx = \frac{1}{2} (\psi(1+n) - \log n). \tag{1.4}$$

This latter result was rediscovered by A.P. Guinand [8] in 1947, and he later found a simpler proof of this result in [9]. In the footnote at the end of his paper [9], Guinand remarks that T.A. Brown had told him that he himself had proved the self-reciprocality of $\psi(1+x) - \log x$ some years ago, and that when he (Brown) communicated the result to G.H. Hardy, Hardy told him that the result was also given by Ramanujan in a progress report to the University of Madras, but was not published elsewhere. However, we cannot find this result in any of the three *Quarterly Reports* that Ramanujan submitted to the University of Madras [3], [4]. Therefore, Hardy's memory was perhaps imperfect; it would appear that he saw (1.4) in the aforementioned manuscript that Watson had copied. On the other hand, the only copy of Ramanujan's *Quarterly Reports* that exists is in Watson's handwriting! It could be that the manuscript on pages 219–220 of [15], which is also in Watson's handwriting, was somehow separated from the original *Quarterly Reports*, and therefore that Hardy was indeed correct in his assertion!

The first equality in (1.3) was rediscovered by Guinand in [8] and appears in a footnote on the last page of his paper [8, p. 18]. It is interesting that Guinand remarks, "This formula also seems to have been overlooked." Here then is one more instance in which a mathematician thought that his or her theorem was new, but unbeknownst to him or her, Ramanujan had beaten them to the punch! We now give Guinand's version of (1.3).

Theorem 1.2. *For any complex z such that $|arg\ z| < \pi$, we have*

$$\sum_{n=1}^{\infty} \left(\frac{\Gamma'}{\Gamma}(nz) - \log nz + \frac{1}{2nz} \right) + \frac{1}{2z}(\gamma - \log 2\pi z)$$
$$= \frac{1}{z} \sum_{n=1}^{\infty} \left(\frac{\Gamma'}{\Gamma}\left(\frac{n}{z}\right) - \log \frac{n}{z} + \frac{z}{2n} \right) + \frac{1}{2}\left(\gamma - \log \frac{2\pi}{z}\right). \tag{1.5}$$

Although not offering a proof of (1.5) in [8], Guinand did remark that it can be obtained by using an appropriate form of Poisson's summation formula, namely the form given in Theorem 1 in [7]. Later Guinand gave another proof of Theorem 1.2 in [9], while also giving extensions of (1.5) involving derivatives of the ψ-function. He also established a finite version of (1.5) in [10]. However, Guinand apparently did not discover the connection of his work with Ramanujan's integral involving Riemann's Ξ-function.

In this paper we first provide a proof of both identities in Theorem 1.1. In Section 4, we construct a second proof of (1.5) along the lines suggested by Guinand in [8]. We can also provide another proof of (1.3) employing both (1.4) and

$$\int_0^\infty \left(\frac{1}{e^{2\pi x}-1}-\frac{1}{2\pi x}\right)e^{-2\pi n x}dx=\frac{1}{2\pi}\left(\log n-\psi(1+n)\right), \tag{1.6}$$

which can be derived from an integral evaluation in [6, p. 377, formula 3.427, no. 7]. However, this proof is similar but slightly more complicated than the first proof that we provide below.

Although the Riemann zeta function appears at various instances throughout Ramanujan's notebooks [13] and lost notebook [15], he only wrote one paper in which the zeta function plays the leading role [12], [14, pp. 72–77]. In fact, a result proved by Ramanujan in [12], namely equation (3.7) in Section 3 below, is a key to proving (1.3). About the integral involving Riemann's Ξ-function in this result, Hardy [11] comments that "the properties of this integral resemble those of one which Mr. Littlewood and I have used, in a paper to be published shortly in the *Acta Mathematica*, to prove that

$$\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+ti\right)\right|^2 dt\sim\frac{2}{\pi}T\log T.\text{"} \tag{1.7}$$

It is also interesting that on a page in the original lost notebook [15, p. 195], Ramanujan defines

$$\phi(x):=\psi(x)+\frac{1}{2x}-\gamma-\log x \tag{1.8}$$

and then concludes that (1.3) is valid. However, with the definition (1.8) of $\phi(x)$, the series in (1.3) do not converge. For a more complete discussion of Ramanujan's incorrect claim, see [2].

2. Preliminary Results

We first collect several well-known theorems that we use in our proof. First, from [5, p. 191], for $t\neq 0$,

$$\sum_{n=1}^{\infty}\frac{1}{t^2+4n^2\pi^2}=\frac{1}{2t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right). \tag{2.1}$$

Second, from [16, p. 251], we find that, for Re $z>0$,

$$\phi(z)=-2\int_0^\infty\frac{tdt}{(t^2+z^2)(e^{2\pi t}-1)}. \tag{2.2}$$

Third, we require Binet's integral for $\log\Gamma(z)$, i.e., for Re $z>0$ [16, p. 249], [6, p. 377, formula 3.427, no. 4],

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log(2\pi)+\int_0^\infty\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-zt}}{t}dt. \tag{2.3}$$

Fourth, from [6, p. 377, formula 3.427, no. 2], we find that

$$\int_0^\infty \left(\frac{1}{1-e^{-x}} - \frac{1}{x} \right) e^{-x} dx = \gamma, \tag{2.4}$$

where γ denotes Euler's constant. Fifth, by Frullani's integral [6, p. 378, formula 3.434, no. 2],

$$\int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{x} dx = \log \frac{\nu}{\mu}, \qquad \mu, \nu > 0. \tag{2.5}$$

3. First Proof of Theorem 1.1

Proof. Our first goal is to establish an integral representation for the far left side of (1.3). Replacing z by $n\alpha$ in (2.2) and summing on n, $1 \le n < \infty$, we find, by absolute convergence, that

$$\begin{aligned}\sum_{n=1}^\infty \phi(n\alpha) &= -2 \sum_{n=1}^\infty \int_0^\infty \frac{tdt}{(t^2+n^2\alpha^2)(e^{2\pi t}-1)} \\ &= \frac{-2}{\alpha^2} \int_0^\infty \frac{t}{(e^{2\pi t}-1)} \sum_{n=1}^\infty \frac{1}{(t/\alpha)^2 + n^2}. \end{aligned} \tag{3.1}$$

Invoking (2.1) in (3.1), we see that

$$\sum_{n=1}^\infty \phi(n\alpha) = -\frac{2\pi}{\alpha} \int_0^\infty \frac{1}{(e^{2\pi t}-1)} \left(\frac{1}{e^{2\pi t/\alpha}-1} - \frac{\alpha}{2\pi t} + \frac{1}{2} \right) dt. \tag{3.2}$$

Next, setting $x = 2\pi t$ in (2.4), we readily find that

$$\gamma = \int_0^\infty \left(\frac{2\pi}{e^{2\pi t}-1} - \frac{e^{-2\pi t}}{t} \right) dt. \tag{3.3}$$

By Frullani's integral (2.5),

$$\int_0^\infty \frac{e^{-t/\alpha} - e^{-2\pi t}}{t} dt = \log \left(\frac{2\pi}{1/\alpha} \right) = \log(2\pi\alpha). \tag{3.4}$$

Combining (3.3) and (3.4), we arrive at

$$\gamma - \log(2\pi\alpha) = \int_0^\infty \left(\frac{2\pi}{e^{2\pi t}-1} - \frac{e^{-t/\alpha}}{t} \right) dt. \tag{3.5}$$

Hence, from (3.2) and (3.5), we deduce that

$$\sqrt{\alpha}\left(\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{n=1}^{\infty}\phi(n\alpha)\right) \tag{3.6}$$
$$=\frac{1}{2\sqrt{\alpha}}\int_0^\infty\left(\frac{2\pi}{e^{2\pi t}-1}-\frac{e^{-t/\alpha}}{t}\right)dt$$
$$-\frac{2\pi}{\sqrt{\alpha}}\int_0^\infty\frac{1}{(e^{2\pi t}-1)}\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}+\frac{1}{2}\right)dt$$
$$=\int_0^\infty\left(\frac{\sqrt{\alpha}}{t(e^{2\pi t}-1)}-\frac{2\pi}{\sqrt{\alpha}(e^{2\pi t/\alpha}-1)(e^{2\pi t}-1)}-\frac{e^{-t/\alpha}}{2t\sqrt{\alpha}}\right)dt.$$

Now from [12, p. 260, eqn. (22)] or [14, p. 77], for n real,

$$\int_0^\infty\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\left(\Xi\left(\frac{1}{2}t\right)\right)^2\frac{\cos nt}{1+t^2}dt$$
$$=\int_0^\infty\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos nt}{1+t^2}dt$$
$$=\pi^{3/2}\int_0^\infty\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)dx. \tag{3.7}$$

Letting $n=\frac{1}{2}\log\alpha$ and $x=2\pi t/\sqrt{\alpha}$ in (3.7), we deduce that

$$-\frac{1}{\pi^{3/2}}\int_0^\infty\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos(\frac{1}{2}t\log\alpha)}{1+t^2}dt \tag{3.8}$$
$$=-\frac{2\pi}{\sqrt{\alpha}}\int_0^\infty\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}\right)\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}\right)dt$$
$$=\int_0^\infty\left(\frac{-2\pi/\sqrt{\alpha}}{(e^{2\pi t/\alpha}-1)(e^{2\pi t}-1)}+\frac{\sqrt{\alpha}}{t(e^{2\pi t}-1)}+\frac{1}{t\sqrt{\alpha}(e^{2\pi t/\alpha}-1)}-\frac{\sqrt{\alpha}}{2\pi t^2}\right)dt.$$

Hence, combining (3.6) and (3.8), in order to prove that the far left side of (1.3) equals the far right side of (1.3), we see that it suffices to show that

$$\int_0^\infty\left(\frac{1}{t\sqrt{\alpha}(e^{2\pi t/\alpha}-1)}-\frac{\sqrt{\alpha}}{2\pi t^2}+\frac{e^{-t/\alpha}}{2t\sqrt{\alpha}}\right)dt$$
$$=\frac{1}{\sqrt{\alpha}}\int_0^\infty\left(\frac{1}{u(e^u-1)}-\frac{1}{u^2}+\frac{e^{-u/(2\pi)}}{2u}\right)du=0, \tag{3.9}$$

where we made the change of variable $u=2\pi t/\alpha$. In fact, more generally, we show that

$$\int_0^\infty\left(\frac{1}{u(e^u-1)}-\frac{1}{u^2}+\frac{e^{-ua}}{2u}\right)du=-\frac{1}{2}\log(2\pi a), \tag{3.10}$$

so that if we set $a=1/(2\pi)$ in (3.10), we deduce (3.9).

Consider the integral, for $t>0$,

$$F(a,t):=\int_0^\infty\left\{\left(\frac{1}{e^u-1}-\frac{1}{u}+\frac{1}{2}\right)\frac{e^{-tu}}{u}+\frac{e^{-ua}-e^{-tu}}{2u}\right\}du$$
$$=\log\Gamma(t)-\left(t-\frac{1}{2}\right)\log t+t-\frac{1}{2}\log(2\pi)+\frac{1}{2}\log\frac{t}{a}, \tag{3.11}$$

where we applied (2.3) and (2.5). Upon the integration of (1.1), it is easily gleaned that, as $t\to 0$,

$$\log\Gamma(t)\sim-\log t-\gamma t,$$

where γ denotes Euler's constant. Using this in (3.11), we find, upon simplification, that, as $t\to 0$,

$$F(a,t)\sim-\gamma t-t\log t+t-\frac{1}{2}\log(2\pi)-\frac{1}{2}\log a.$$

Hence,

$$\lim_{t\to 0}F(a,t)=-\frac{1}{2}\log(2\pi a). \tag{3.12}$$

Letting t approach 0 in (3.11), taking the limit under the integral sign on the right-hand side using Lebesgue's dominated convergence theorem, and employing (3.12), we immediately deduce (3.10). As previously discussed, this is sufficient to prove the equality of the first and third expressions in (1.3), namely,

$$\sqrt{\alpha}\left\{\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{n=1}^\infty\phi(n\alpha)\right\}=-\frac{1}{\pi^{3/2}}\int_0^\infty\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt. \tag{3.13}$$

Lastly, using (3.13) with α replaced by β and employing the relation $\alpha\beta=1$, we conclude that

$$\sqrt{\beta}\left\{\frac{\gamma-\log(2\pi\beta)}{2\beta}+\sum_{n=1}^\infty\phi(n\beta)\right\}$$
$$=-\frac{1}{\pi^{3/2}}\int_0^\infty\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\beta\right)}{1+t^2}dt$$
$$=-\frac{1}{\pi^{3/2}}\int_0^\infty\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log(1/\alpha)\right)}{1+t^2}dt$$
$$=-\frac{1}{\pi^{3/2}}\int_0^\infty\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt.$$

Hence, the equality of the second and third expressions in (1.3) has been demonstrated, and so the proof is complete. □

4. Second Proof of (1.3)

In this section we give our second proof of the first identity in (1.3) using Guinand's generalization of Poisson's summation formula in [7]. We emphasize that this route does not take us to the integral involving Riemann's Ξ-function in the second identity of (1.3). First, we reproduce the needed version of the Poisson summation formula from Theorem 1 in [7].

Theorem 4.1. *If $f(x)$ is an integral, tends to zero at infinity, and $xf'(x)$ belongs to $L^p(0,\infty)$, $(1<p\le 2)$, then*

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} f(n) - \int_0^N f(t)dt\right) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} g(n) - \int_0^N g(t)dt\right), \tag{4.1}$$

where

$$g(x) = 2\int_0^{\to\infty} f(t)\cos(2\pi x t)dt. \tag{4.2}$$

Next, we state a lemma[1] that will subsequently be used in our proof of (1.3).

Lemma 4.2.

$$\int_0^\infty \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t\right) dt = \frac{1}{2}\log 2\pi. \tag{4.3}$$

Proof. Let I denote the integral on the left-hand side. Then,

$$\begin{aligned} I &= \int_0^\infty \frac{d}{dt}\left(\ln\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}}\right) dt \\ &= \lim_{t\to\infty}\ln\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} - \lim_{t\to 0}\ln\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} \\ &= \ln\lim_{t\to\infty}\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} - \ln\left(\lim_{t\to 0} e^t\Gamma(t+1)\right) - \lim_{t\to 0} t\ln t - \lim_{t\to 0}\frac{1}{2}\ln(t+1) \\ &= \ln\lim_{t\to\infty}\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}}. \end{aligned} \tag{4.4}$$

Next, Stirling's formula [6, p. 945, formula 8.327] tells us that,

$$\Gamma(z) \sim z^{z-1/2}e^{-z}\sqrt{2\pi}, \tag{4.5}$$

as $|z|\to\infty$ and $|\arg z| \le \pi-\delta$, where $0<\delta<\pi$. Hence, employing (4.5), we find that

$$\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} \sim \left(1+\frac{1}{t}\right)^t \frac{\sqrt{2\pi}}{e}, \tag{4.6}$$

so that

$$\lim_{t\to\infty}\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} = \sqrt{2\pi}. \tag{4.7}$$

Thus from (4.4) and (4.7), we conclude that

$$I = \frac{1}{2}\ln 2\pi. \tag{4.8}$$

□

[1] The authors are indebted to M. L. Glasser for the proof of this lemma. The authors' original proof of this lemma was substantially longer than Glasser's given here.

Now we are ready to give our second proof of (1.3). We first prove it for real $z > 0$. Let

$$f(x) = \psi(xz+1) - \log xz. \tag{4.9}$$

We show that $f(x)$ satisfies the hypotheses of Theorem 4.1. From (1.4), it is true that $f(x)$ is an integral. Next, we need two formulas for $\psi(x)$.

First, from [1, p. 259, formula 6.3.18] for $|\arg z| < \pi$, as $z \to \infty$,

$$\psi(z) \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots . \tag{4.10}$$

Second, we use the fact from [16, p. 250] that

$$\psi^{'}(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}. \tag{4.11}$$

From (4.10), we have

$$f(x) \sim \frac{1}{2xz} - \frac{1}{12x^2z^2} + \frac{1}{120x^4z^4} - \frac{1}{252x^6z^6} + \cdots , \tag{4.12}$$

so that

$$\lim_{x\to\infty} f(x) = 0. \tag{4.13}$$

Next, we show that $xf^{'}(x)$ belongs to $L^p(0,\infty)$ for some p such that $1 < p \leq 2$.

Using (4.10), we find that

$$xf^{'}(x) = xz\psi^{'}(xz) - \frac{1}{xz} - 1 \sim -\frac{1}{2xz}, \tag{4.14}$$

so that $|xf^{'}(x)|^p \sim \frac{1}{2^p z^p x^p}$. Now $p > 1$ implies that $xf^{'}(x)$ is locally integrable near ∞. Also, using (4.11), we have

$$\begin{aligned}\lim_{x\to 0} xf^{'}(x) &= \lim_{x\to 0}\left(xz\sum_{n=0}^{\infty}\frac{1}{(xz+n)^2} - \frac{1}{xz} - 1\right)\\ &= \lim_{x\to 0}\left(xz\sum_{n=1}^{\infty}\frac{1}{(xz+n)^2} - 1\right)\\ &= -1. \end{aligned} \tag{4.15}$$

This proves that $xf^{'}(x)$ is locally integrable near 0. Hence it is true that $xf^{'}(x)$ belongs to $L^p(0,\infty)$ for some p such that $1 < p \leq 2$.

Now from (4.2) and (4.9), we find that

$$g(x) = 2\int_0^{\infty} (\psi(tz+1) - \log tz)\cos(2\pi xt)\, dt.$$

Employing a change of variable $y = tz$ and using (1.4), we find that

$$\begin{aligned} g(x) &= \frac{2}{z}\int_0^{\infty} (\psi(y+1) - \log y)\cos(2\pi xy/z)\, dy\\ &= \frac{1}{z}\left(\psi\left(\frac{x}{z}+1\right) - \log\left(\frac{x}{z}\right)\right). \end{aligned} \tag{4.16}$$

Substituting the expressions for $f(x)$ and $g(x)$ from (4.9) and (4.16), respectively, in (4.1), we find that

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}(\psi(nz+1)-\log nz)-\int_0^N(\psi(tz+1)-\log tz)dt\right)$$
$$=\frac{1}{z}\left[\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\psi\left(\frac{n}{z}+1\right)-\log\frac{n}{z}\right)-\int_0^N\left(\psi\left(\frac{t}{z}+1\right)-\log\frac{t}{z}\right)dt\right)\right]. \tag{4.17}$$

Thus,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{\Gamma'}{\Gamma}(nz)+\frac{1}{2nz}-\log nz\right)+\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)dt\right)$$
$$=\frac{1}{z}\left[\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{\Gamma'}{\Gamma}\left(\frac{n}{z}\right)+\frac{z}{2n}-\log\frac{n}{z}\right)+\sum_{n=1}^{N}\frac{z}{2n}-\int_0^N\left(\psi\left(\frac{t}{z}+1\right)-\log\frac{t}{z}\right)dt\right)\right]. \tag{4.18}$$

Now if we can show that

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)dt\right)=\frac{\gamma-\log 2\pi z}{2z}, \tag{4.19}$$

then replacing z by $1/z$ in (4.19) will give us

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{z}{2n}-\int_0^N\left(\psi\left(\frac{t}{z}+1\right)-\log\frac{t}{z}\right)dt\right)=\frac{z(\gamma-\log(2\pi/z))}{2}. \tag{4.20}$$

Then substituting (4.19) and (4.20) in (4.18) will complete the proof of the theorem. So our goal now is to prove (4.19).

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)dt\right)$$
$$=\lim_{N\to\infty}\left(\frac{1}{2z}\left(\sum_{n=1}^{N}\frac{1}{n}-\log N\right)+\frac{\log N}{2z}-\int_0^N(\psi(tz+1)-\log tz)dt\right)$$
$$=\frac{\gamma}{2z}+\lim_{N\to\infty}\left(-\frac{\log z}{2z}+\frac{\log Nz}{2z}-\int_0^N(\psi(tz+1)-\log tz)dt\right)$$
$$=\frac{\gamma}{2z}-\frac{\log z}{2z}+\lim_{N\to\infty}\left(\frac{\log(Nz+1)}{2z}-\frac{1}{z}\int_0^{Nz}(\psi(t+1)-\log t)dt-\frac{1}{2z}\log\left(1+\frac{1}{Nz}\right)\right)$$
$$=\frac{\gamma}{2z}-\frac{\log z}{2z}+\frac{1}{z}\lim_{N\to\infty}\left(\frac{\log(Nz+1)}{2}-\int_0^{Nz}(\psi(t+1)-\log t)dt\right)$$
$$=\frac{\gamma}{2z}-\frac{\log z}{2z}+\frac{1}{z}\lim_{N\to\infty}\left(\frac{1}{2}\int_0^{Nz}\frac{1}{t+1}dt-\int_0^{Nz}(\psi(t+1)-\log t)dt\right)$$

$$
\begin{aligned}
&= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{1}{z}\lim_{N\to\infty}\int_0^{Nz}\left(\psi(t+1) - \frac{1}{2(t+1)} - \log t\right)dt\\
&= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{1}{z}\int_0^{\infty}\left(\psi(t+1) - \frac{1}{2(t+1)} - \log t\right)dt\\
&= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{\log 2\pi}{2z}\\
&= \frac{\gamma - \log 2\pi z}{2z}, \qquad (4.21)
\end{aligned}
$$

where in the antepenultimate line, we have made use of Lemma 4.2. This completes the proof of (4.19) and hence the proof of Theorem 1.2 for real $z > 0$. But both sides of (1.5) are analytic for $|\arg z| < \pi$. Hence by analytic continuation, the theorem is true for all complex z such that $|\arg z| < \pi$.

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